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A THEORETICAL INVESTIGATION OF FINITE AMPLITUDE STANDING WAVES IN RIGID WALLED CAVITIES

PAUL GRAY RUFF

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A THEORETICAL INVESTIGATION OF
FINITE AMPLITUDE STANDING WAVES
IN RIGID WALLED CAVITIES

by

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ABSTRACT

The Coppens-Sanders perturbation solution for the one-dimensional non-linear acoustic wave equation with dissipative term describing the viscous and thermal energy losses encountered in a rigid walled, closed tube with large length-to-diameter ratio was extended to include sixth order terms. The solution was then investigated to determine the region of validity. Computer programs were written to evaluate and graph the resulting waveforms. Available experimental results were compared with the theoretical predictions and good correlation was found to exist in the region of low Mach numbers. This agreement was found to gradually deteriorate as the Mach number was increased. A Fourier synthesis approach is also presented and the leading terms of the first ten harmonics are derived.

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LIST OF SYMBOLS

a = Lagrangian coordinate measured from piston

A_o = (peak) acceleration amplitude of piston

B/A = parameter of nonlinearity = $\rho_o (d^2 P/\partial\rho^2)/(dP/d\rho)$ @ $\rho = \rho_o$

c = phase velocity in the tube

c_o = $(dP/d\rho)^{1/2}$ @ $\rho = \rho_o$

k = ω/c

$\underset{\sim}{k}$ = $k - i\alpha$

K = $m\pi/L$

L = Lagrangian coordinate of rigid end of tube

m = normal mode of tube most strongly excited by input frequency

M = U_{11}/c_o peak Mach number of first-order solution

p, P_o = acoustic pressure, equilibrium pressure

p_{nj} = j-th frequency component of the n-th order perturbation pressure

P_{11} = (infinitesimal) pressure amplitude at the rigid end of the tube

H_n = resonance parameter (see Eq. 36)

u = particle velocity

u_n = particle velocity of the n-th order perturbation solution

u_{nj} = j-th frequency component of u_n

α = infinitesimal-amplitude attenuation constant

β = $1 + \frac{1}{2}(B/A)$

γ = c_p/c_v = ratio of specific heats

δ = dissipation parameter (see Eq. 19)

δ_j = value of δ for (angular) frequency ω_j

θ_n = phase parameter

ξ = particle displacement

ρ, ρ_o = instantaneous density, equilibrium density

ϕ_n = forcing term (see Eq. 16)

ω = $2\pi f$ = (angular) driving frequency

ω_r = infinitesimal-amplitude resonance frequency

$\Delta\omega = \omega - \omega_r$

$D = (\rho_o c_o^2)^{-1} \frac{\partial}{\partial t} \mathcal{F}$

D_j = D for frequency ω_j

$\mathcal{F}$ = operator for body forces (see Eq. 9)

$\Box_L^2 = \frac{\partial^2}{\partial a^2} - \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2}$ = D'Alambertian in Lagrangian coordinates (one spatial dimension)

V_A = (peak) amplitude of voltage output of accelerometer

V_M = (peak) amplitude of voltage output of microphone

$S_A = V_A/A_o$ = accelerometer sensitivity

$S_M = V_M/P$ = microphone sensitivity

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1. INTRODUCTION

The thermodynamic equation of state of a fluid for reversible acoustic processes can be written as

$$p = \frac{A}{\rho_0}(\rho-\rho_0) + \frac{B}{2\rho_0^2}(\rho-\rho_0)^2 \tag{1.1}$$

or, for an ideal gas,

$$\frac{p+p_0}{p_0} = (\rho/\rho_0)^\gamma \tag{1.2}$$

where γ is the ratio of specific heats. The continuity equation for one-dimensional wave propagation in Lagrangian coordinates is

$$\rho\left(1+\frac{\partial\xi}{\partial a}\right) = \rho_0 . \tag{1.3}$$

The force equation in the absence of dissipative mechanisms has the form

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial a} . \tag{1.4}$$

It is possible by straightforward combination to obtain the equation

$$\frac{\partial^2\xi}{\partial t^2} - c_0^2\left(1+\frac{\partial\xi}{\partial a}\right)^{-2}\left[1 - \left(\frac{B}{A}\right)\frac{\partial\xi}{\partial a}\left(1+\frac{\partial\xi}{\partial a}\right)^{-1}\right]\frac{\partial^2\xi}{\partial a^2} = 0 \tag{1.5}$$

for a fluid obeying Eq. 1.1 or

$$\frac{\partial^2\xi}{\partial t^2} - c_0^2\left(1+\frac{\partial\xi}{\partial a}\right)^{-\gamma-1}\frac{\partial^2\xi}{\partial a^2} = 0 \tag{1.6}$$

for an ideal gas.

If interest is restricted to acoustic processes for which $\partial\xi/\partial a \ll 1$, Eqs. 1.5 and 1.6 may be approximated by

$$\Box^2 \xi = \beta \frac{\partial}{\partial a}\left(\frac{\partial \xi}{\partial a}\right)^2 \tag{1.7}$$

which (upon differentiation with respect to time) becomes the non-dissipative wave equation

$$\Box^2 u = \beta \frac{\partial^2}{\partial a \partial t}\left(\frac{\partial \xi}{\partial a}\right)^2 \tag{1.8}$$

where

$$\beta = 1 + \tfrac{1}{2}\left(B/A\right) = \frac{\gamma+1}{2} \tag{1.9}$$

$$\Box^2 = \frac{\partial^2}{\partial a^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}.$$

A more general form of Eq. 1.4 is

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial a} + \mathcal{F}_L u \tag{1.10}$$

where $\mathcal{F}_L$ is an operator which generates all forces, other than that arising from the gradient of the pressure field, which are active in the system of interest [4]. If Eq. 1.10 is used instead of Eq. 1.4, Eq. 1.7 becomes

$$\Box^2 \xi + \frac{1}{\rho_0 c_0^2}\mathcal{F}_L u \doteq \beta \frac{\partial}{\partial a}\left(\frac{\partial \xi}{\partial a}\right)^2 \tag{1.11}$$

In the Navier-Stokes case

$$\mathcal{F}_L = \left(\tfrac{4}{3}\eta + \eta_B\right)\frac{\partial^2}{\partial a^2} \tag{1.12}$$

where η and η_B are the shear and bulk viscosity coefficients, respectively.

An important contribution to the solution of Eq. 1.11 for the case of Eq. 1.12 was made by Fay [5]. He investigated the changes in the steepness of the wavefront in a periodic, finite-amplitude plane wave of infinite extent in a viscous medium. By means of Fourier analysis techniques he was able to show that the nonlinearity of the pressure density relationship results in accumulating distortion of the waveform with shifts of energy from the lower to the higher frequency components of the wave. Because the classical Navier-Stokes absorption coefficient is proportional to the square of the frequency, the higher frequency components are attenuated more rapidly. Fay's solution is valid in a region far from the source where the rates of harmonic growth due to nonlinearity and the rates of attenuation due to viscosity are tending to balance leading to a waveform of relatively stable shape over distances corresponding to many wavelengths.

Fox and Wallace [1] investigated the same problem as Fay using a graphical analysis technique. Their result is essentially the same as Fay's. Keck and Beyer [6] developed a perturbation analysis technique for periodic plane progressive waves of infinite extent in a viscous medium with which they generate terms through sixth order. The wave equation is the same as that treated by Fay, but the solution is valid only near the source.

Weston [20] has presented a linearized wave equation for the propagation of monofrequency sound in tubes. He assumes that the wavefronts are planar except near the walls. The viscous and heat-conduction losses in the boundary layer are assumed to be the dominant dissipation mechanisms in the fluid. Except in this boundary layer, the wavefronts are independent of a radial coordinate. This suggests that the boundary layer loss can be replaced by an equivalent absorptive process active throughout the volume of the cavity [4]. More will be said about this later.

Saenger and Hudson [14] present a simple description of periodic shocks at resonance using a two part solution. Their solution is based on the assumption of the applicability of the linearized acoustic equations everywhere except in the region of the shock where the Rankine-Hugoniot shock conditions are assumed. They consider the case in which the piston oscillates at the fundamental frequency. Compressive viscosity is ignored but the effects of shear viscosity and heat conduction in the boundary layer are considered. Their solution remains finite only because of these considerations.

Betchov [13] also postulates the existence of the shock wave and constructs a solution at resonance based on a continuous and a discontinuous part. For an inviscid fluid the amplitude at resonance is found to be finite and to be determined by nonlinear effects. The effect of wall friction is discussed and it is suggested that this could modify the solution significantly.

Weiss [7] has applied finite-difference techniques to the nonlinear inviscid acoustic equations and has shown that repeated reflections at the rigid boundaries tend to promote the development of a discontinuity in the velocity profile. The absence of any dissipative mechanisms, however, eliminates the possibility of any steady-state standing wave patterns.

While most investigations have been confined to traveling waves, a few have considered standing waves.

Solutions of the nonlinear wave equation have been obtained for the case of finite-amplitude standing waves in a closed tube, where the tube is driven at one end by a piston source. This work was done by Keller [21] using the Lagrangian formulation and assigning an unrealistic value of -1 to the adiabatic exponent γ. Due to neglect of any dissipative mechanisms, his expression for the particle velocity becomes infinite if the piston frequency equals a natural frequency of the tube.

A detailed theoretical analysis based on the nonlinear acoustic equations including the effects of compressive viscosity and shear viscosity in the boundary layer has been done by Chester [8]. He successfully predicted asymmetries in the finite-amplitude standing waves. Unfortunately his results are not easily compared to experimental results.

Coppens and Sanders [4] developed an extension of the Keck-Beyer perturbation approach wherein wall losses as well as bulk losses are included. Their solution is applied to the case of finite-amplitudes standing waves in rigid walled cavities. The results yield information concerning the amplitudes and phases of the Fourier components of the waveform, and indicate the importance of the type of absorptive process on the resulting waveform.

The purpose of this thesis is to extend the Coppens-Sanders perturbation approach. In part this is accomplished by a direct application of their method to obtain sixth-order terms. Information concerning the response of the cylindrical cavity to the sinusoidal excitation is extracted from this solution by some algebraic manipulations.

A Fourier synthesis approach to the problem is presented and from it the leading terms of the first ten harmonics are derived. Information similar to that obtained from the perturbation approach is acquired and both results are compared to the experimental results obtained by Beech [19].

2. THEORETICAL DEVELOPMENT

In this section both a perturbation approach and a Fourier-synthesis approach to the solution of the one-dimensional non-linear, dissipative acoustic wave equation are presented. These approaches are formulated for a rigid-walled, closed, cylindrical cavity with large length-to-diameter ratio. The solutions obtained are for finite-amplitude standing waves. Manipulations are also presented which enable each approach to yield resonance response information which is convenient for comparison with experimental results.

Perturbation Approach

Assume a perturbation series $\xi = \sum_n \xi_n$ such that the n-th term is of order n-1 in the Mach number. Substituting this into Eq. 1.8 and collecting terms of equal order yields the set of equations

$$\left(\Box_L^2 + D\right) u_n = \phi_n \tag{2.1}$$

where

$$\phi_n = \begin{cases} \beta \dfrac{\partial^2}{\partial a \partial t} \displaystyle\sum_{i=1}^{n-1} \dfrac{\partial \xi_i}{\partial a} \dfrac{\partial \xi_{n-i}}{\partial a}, & n > 1 \\ 0 & n < 1 \end{cases} \tag{2.2}$$

If $u_n = \sum_j u_{nj}$, where the summation over j is understood to encompass only those frequencies included in each ϕ_n, Eq. 2.1 becomes

$$\sum_j \left(\Box_L^2 + D_j\right) u_{nj} = \phi_n \tag{2.3}$$

In order to utilize this formulation, it is first necessary to obtain the form of Dj. For the case of finite amplitude sound in a duct, Weston [20] has presented a wave equation valid for the propagation of monofrequency sound. Weston's equation may be written in Eulerian form as

$$\Box_E^2 \underset{\sim}{u} + \delta(i-1)\,\frac{\partial \underset{\sim}{u}}{\partial x} = 0 \tag{2.4}$$

where

$$\delta = \left(\frac{G}{S}\right)(2\omega)^{-1/2}\left[\nu^{1/2} + (1-\gamma^{-1})\,\nu'^{\,1/2}\right] \tag{2.5}$$

and

S = (uniform) cross-sectional area
G = perimeter of area S
ω = (angular) frequency of the propagating waveform
ν = kinematic shear viscosity of the fluid
ν' = thermometric conductivity of the walls
$i = \sqrt{-1}$

This may be rewritten as

$$\Box_E^2 u + \left(\frac{\delta}{\omega}\right)\frac{\partial^3 u}{\partial x^2 \partial t} - \delta\,\frac{\partial^2 u}{\partial x^2} = 0 \tag{2.6}$$

When dissipation processes are included, it is usually assumed, explicitly or tacitly, that they are weak. Under this assumption, and for small Mach number, the derivatives ($\partial/\partial a$)t and ($\partial/\partial x$)t may be interchanged. This also holds for the pair ($\partial/\partial t$)a and ($\partial/\partial t$)x. Thus Eq. 2.3 must be solved with

$$D_j \doteq \left(\frac{\delta_j}{\omega_j}\right)\frac{\partial^3}{\partial a^2 \partial t} - \delta_j\,\frac{\partial^2}{\partial a^2} \tag{2.7}$$

where δ_j represents the value of δ for a frequency ω_j.

The final result is the approximate wave equation

$$\sum_j \left(\Box^2 + \frac{\delta_j}{\omega_j} \frac{\partial^3}{\partial a^2 \partial t} - \delta_j \frac{\partial^2}{\partial a^2} \right) u_{nj} = \beta \frac{\partial^2}{\partial a \partial t} \sum_{i=1}^{n-1} \frac{\partial \xi_i}{\partial a} \frac{\partial \xi_{n-i}}{\partial a} \tag{2.8}$$

Eq. 2.8 was solved [4] for a uniform tube of length L excited at end a = 0 by a piston driven with constant acceleration A_o at a frequency ω, and terminated at a = L by a rigid cap. In this solution, the boundary conditions were given by

$$\begin{aligned} u_{11}(0,t) &= (A_0/\omega) \sin(\omega t - \theta_1) \\ u_{11}(L,t) &= u_{nj}(0,t) = u_{nj}(L,t) = 0, \quad n>1 \end{aligned} \tag{2.9}$$

where θ will be defined later.

The lowest order solution of Eq. 2.8 is

$$u_1 = u_{11} = \Re(\underset{\sim}{u}_{11}) \tag{2.10}$$

where

$$\underset{\sim}{u}_{11} = \frac{A_0}{i\omega} e^{i(\omega t - \theta_1)} \frac{e^{i\underset{\sim}{k}(L-a)} - e^{-i\underset{\sim}{k}(L-a)}}{e^{i\underset{\sim}{k}L} - e^{-i\underset{\sim}{k}L}} \tag{2.11}$$

and $k = k - i\alpha$ is the complex propagation constant.

The solution for $\mathbf{u}_{11}$ may be rewritten as

$$\begin{aligned} u_{11} = U_{11} \Big\{ &[\cosh \alpha(L-a) \sin k(L-a)] \cos \omega t \\ &+ [\sinh \alpha(L-a) \cos k(L-a)] \sin \omega t \Big\} \end{aligned} \tag{2.12}$$

where

$$U_{11} = \left(\frac{A_0}{\omega}\right) \left(\sinh^2 \alpha L + \sin^2 kL \right)^{-1/2} \tag{2.13}$$

and

$$\theta_1 \equiv \tan^{-1}\left(-\frac{\tan kL}{\tanh \alpha L}\right) \tag{2.14}$$

Eqs. 2.12 and 2.13 will be rewritten in terms of ω_r, the resonant frequency which maximizes the pressure amplitude p_{11}, associated with U_{11}, at the rigid end of the cavity.

$$\begin{aligned} \frac{\omega_r}{c} &\doteq \frac{m\pi}{L} - \frac{\alpha^2 L}{m\pi}, \quad m = 1, 2, \cdots \\ \omega &= \omega_r + \Delta\omega \end{aligned} \tag{2.15}$$

The integer m represents the normal mode most strongly excited by the input frequency. For $\alpha L \ll 1$ and $\Delta\omega \ll \frac{c}{L}$, U_{11} and θ_1 are

$$U_{11} \doteq \frac{2A_0}{m\pi\delta_1\omega_r}\cos\theta_1 \tag{2.16}$$

$$\theta_1 \doteq \tan^{-1}\left(-\frac{2\Delta\omega}{\delta_1\omega_r}\right). \tag{2.17}$$

From this point on, higher-order solutions are obtained by first making the approximation

$$u_{11}/U_{11} \doteq \sin K(L-a)\cos\omega t \tag{2.18}$$

which is Eq. 2.12 with terms of α/k omitted. The procedure (presented in Appendix A), is then merely an iteration process.

In view of later discussions it is advantageous to point out that Eq. 2.18 may be rewritten in terms of the associated acoustic pressure by merely making the substitutions

$$\begin{aligned} u &\longrightarrow p/\rho_0 c_0 & \qquad U_{11} &\longrightarrow P_{11}/\rho_0 c_0 \\ \sin &\longrightarrow \cos & \qquad \cos &\longrightarrow \sin \end{aligned} \tag{2.19}$$

This is equivalent to ignoring terms of order α/k in Eq. 1.10, and converts the calculated variable from velocity to pressure, a more commonly measured quantity.

In terms of the pressure, the solution presented by Coppens and Sanders is

$$
\begin{aligned}
p/P_{11} \doteq{} & \cos K(L-a)\Big\{\sin\omega t - \tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)^2(1,2)_1 \\
& \quad + \tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)^4\Big[(1,2,2,3)_1 - \tfrac{1}{2}(1,2,-2,3)_1 \\
& \quad + (1,1,2,2)_1 + \tfrac{1}{2}(1,1,2,-2)_1\Big] + \cdots\Big\} \\
& + \cos 2K(L-a)\Big\{\tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)(2)_2 - \tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)^3\Big[(2,2,3)_2 \\
& \quad + (1,2,2)_2\Big] + \cdots\Big\} \\
& + \cos 3K(L-a)\Big\{\tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)^2(2,3)_3 - \tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)^4\Big[(2,3,3,4)_3 \\
& \quad + \tfrac{1}{4}(2,2,3,4)_3 + (2,2,3,3)_3 \\
& \quad + \tfrac{3}{2}(1,2,2,3)_3\Big] + \cdots\Big\} \\
& + \cos 4K(L-a)\Big\{\tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)^3\Big[(2,3,4)_4 + \tfrac{1}{4}(2,2,4)_4\Big] + \cdots\Big\} \\
& + \cos 5K(L-a)\Big\{\tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)^4\Big[(2,3,4,5)_5 \\
& \quad + \tfrac{1}{4}(2,2,4,5)_5 + \tfrac{1}{2}(2,2,3,5)_5\Big] + \cdots\Big\} \\
& + \cdots
\end{aligned}
\tag{2.20}
$$

where

$$(a,b)_g = \frac{1}{H_a H_b} \sin \left(g\omega t + \theta_a + \theta_b \right) \tag{2.21}$$

$$H_n = \left[\left(\delta_n - \delta_1 + 2\Delta\omega/\omega_p \right)^2 + \delta_n^{\,2} \right]^{1/2} \tag{2.22}$$

$$\theta_n = -\cot^{-1}\left[\delta_n / \delta_n - \delta_1 + 2\Delta\omega/\omega_p \right], \; n = 1, 2, \cdots \tag{2.23}$$

$$M = U_{11}/c_0 \tag{2.24}$$

and M is the peak Mach number of the first order solution.

Using the same scheme, the sixth order terms were derived. The solution for p_6 which is to be added to Eq. 2.20 is

$$\begin{aligned} p_6/P_{11} = \cos 2K(L-a) \Big\{ \frac{1}{2}\left(\frac{M\beta}{2}\right)^5 \Big[& (2,2,3,3,4)_2 \\ & + \frac{1}{4}(2,2,2,3,4)_2 + (2,2,2,3,3)_2 \\ & + \frac{5}{2}(1,2,2,2,3)_2 + (1,1,2,2,2)_2 \\ & - \frac{1}{2}(1,2,2,-2,3)_2 + \frac{1}{2}(1,1,2,-2,2)_2 \\ & - \frac{1}{2}(2,2,-2,3,4)_2 - \frac{1}{8}(2,2,2,-2,4)_2 \\ & + \frac{1}{2}(-1,2,-2,2,3)_2 + \frac{1}{4}(1,1,2,2,2)_2 \Big] + \cdots \Big\} \end{aligned} \tag{2.25}$$

$$
\begin{aligned}
-\cos 4K(L-a)\Big\{\tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)^{5}\Big[&(2,3,4,4,5)_4 \\
&+\tfrac{1}{4}(2,2,4,4,5)_4+\tfrac{1}{2}(2,2,3,4,5)_4 \\
&+(2,3,3,4,4)_4+\tfrac{1}{4}(2,2,3,4,4)_4 \\
&+(2,2,3,3,4)_4+(1,2,2,3,4)_4 \\
&+\tfrac{1}{2}(1,2,2,3,4)_4+\tfrac{1}{2}(2,2,2,3,4)_4 \\
&+\tfrac{1}{2}(1,2,2,2,4)_4+\tfrac{1}{2}(1,2,2,3,4)_4\Big]+\cdots\Big\} \\
+\cos 6K(L-a)\Big\{\tfrac{1}{2}\left(\tfrac{M\beta}{2}\right)^{5}\Big[&(2,3,4,5,6)_6 \\
&+\tfrac{1}{4}(2,2,4,5,6)_6+\tfrac{1}{2}(2,2,3,5,6)_6 \\
&+\tfrac{1}{2}(2,2,3,4,6)_6+\tfrac{1}{8}(2,2,2,4,6)_6 \\
&+\tfrac{1}{4}(2,2,3,3,6)_6\Big]+\cdots\Big\}
\end{aligned}
$$

The entire perturbation solution herein developed, which is the sum of Eqs. 2.20 and 2.25, will be discussed in some detail after the Fourier synthesis approach has been presented.

Fourier Synthesis

Beginning with the approximate wave equation Eq. 2.8 and assuming the total solution is composed of the known classical solution ξ_c satisfying

$$\square^2 \xi_c + \frac{\delta_j}{\omega_j}\frac{\partial^3 \xi_c}{\partial a^2 \partial t} - \delta_j \frac{\partial^2 \xi_c}{\partial a^2} = 0 \tag{2.26}$$

and some other portion ξ^0 defined by

$$\xi = \xi_c + \xi^0 \tag{2.27}$$

then

$$\square^2 \xi^0 + \frac{\delta_j}{\omega_j}\frac{\partial^3 \xi^0}{\partial a^2 \partial t} - \delta_j \frac{\partial^2 \xi^0}{\partial a^2} \doteq \frac{\gamma+1}{2}\frac{\partial}{\partial a}\left(\frac{\partial \xi_c}{\partial a} + \frac{\partial \xi^0}{\partial a}\right)^2 \tag{2.28}$$

Define the phase velocity c_j to be $c_j^2 = c_0^2(1-\delta_j)$ and Eq. 2.28 may be written

$$\left(\frac{c_j}{c_0}\right)^2 \frac{\partial^2 \xi^0}{\partial a^2} - \frac{1}{c_0^2}\frac{\partial^2 \xi^0}{\partial t^2} + \left(\frac{\delta_j}{\omega_j}\right)\frac{\partial^3 \xi^0}{\partial a^2 \partial t} = \frac{\gamma+1}{2}\frac{\partial}{\partial a}\left(\frac{\partial \xi_c}{\partial a} + \frac{\partial \xi^0}{\partial a}\right)^2 \tag{2.29}$$

Assume ξ may be written as a Fourier series

$$\frac{\partial \xi^0}{\partial a} \doteq \sum_{n=0}^{\infty} X_n^0 \cos nK(L-a)\cos(n\omega t + \phi_n^0) \tag{2.30}$$

and

$$\frac{\partial \xi_c}{\partial a} = X_c \cos K(L-a) \cos \omega t \tag{2.31}$$

and let

$$\left(\frac{\partial \xi}{\partial a}\right)^2 \doteq \sum_{n=0}^{\infty}{}' A_n(a) \cos(n\omega t + \Gamma_n) \tag{2.32}$$

where A_n and Γ_n will be obtained later. Combination of Eqs. 2.29 through 2.32 yields

$$\begin{aligned} &\left[\left(\frac{c_n}{c_0}\right)^2 - \left(\frac{\omega}{Kc_0}\right)^2\right] X_n^0 \sin nK(L-a) \cos(n\omega t + \phi_n^0) \\ &\qquad - \delta_n X_n^0 \sin nK(L-a) \sin(n\omega t + \phi_n^0) \\ &\qquad\qquad \doteq \frac{\gamma+1}{2} \frac{1}{nK} A_n'(a) \cos(n\omega t + \Gamma_n) \end{aligned} \tag{2.33}$$

We impose the conditions that

$$A_n(a) = B_n \cos nK(L-a) \tag{2.34}$$

and

$$A_n'(a) \propto nK \sin nK(L-a)$$

so that Eq. 2.33 becomes

$$\begin{aligned} &\left\{\left[\left(\frac{c_n}{c_0}\right)^2 - \left(\frac{\omega}{c_0 K}\right)^2\right] \cos(n\omega t + \phi_n^0) - \delta_n \sin(n\omega t + \phi_n^0)\right\} X_n^0 \\ &\qquad = \frac{\gamma+1}{2} B_n \cos(n\omega t + \Gamma_n) \end{aligned} \tag{2.35}$$

Introduction of the phase angles θ_n of Eq. 2.17 allows Eq. 2.35 to be written

$$-X_n^0 \sin(n\omega t + \phi_n^0 - \theta_n) = \frac{\gamma+1}{2} \frac{\cos\theta_n}{\delta_n} B_n \cos(n\omega t + \Gamma_n) \tag{2.36}$$

In order to simplify the manipulations to obtain B_n and Γ_n define

$$X_1 \cos K(L-a)\cos(\omega t+\phi_1) = X_1^\circ \cos K(L-a)\cos(\omega t+\phi_1^\circ) + X_c \cos K(L-a)\cos \omega t \qquad (2.37)$$

and

$$X_n \cos nK(L-a)\cos(n\omega t+\phi_n) = X_n^\circ \cos nK(L-a)\cos(n\omega t+\phi_n^\circ)\,, \quad n\neq 1 \qquad (2.38)$$

Thus

$$\frac{\partial \xi}{\partial a} \doteq \sum_{n=1}^{\infty} X_n \cos nK(L-a)\cos(n\omega t+\phi_n) \qquad (2.39)$$

and

$$2B_n \cos(n\omega t+\Gamma_n) \doteq \sum_{j=1}^{\infty} X_j X_{n+j}\cos(n\omega t+\phi_{n+j}-\phi_j) + \frac{1}{2}\sum_{j=1}^{n-1} X_j X_{n-j}\cos(n\omega t+\phi_{n-j}+\phi_j) \qquad (2.40)$$

Combining Eqs. 2.40 and 2.36 gives

$$-\frac{4}{\gamma+1}\frac{\delta_n}{\cos\theta_n} X_n^\circ \sin(n\omega t+\phi_n^\circ-\theta_n) = \sum_{j=1}^{\infty} X_j X_{n+j}\cos(n\omega t+\phi_{n+j}-\phi_j) + \frac{1}{2}\sum_{j=1}^{n-1} X_j X_{n-j}\cos(n\omega t+\phi_{n-j}+\phi_j) \qquad (2.41)$$

It is convenient to shift to complex notation at this time. Let

$$X_1 \cos\phi_1 = X_1^\circ \cos\phi_1^\circ + X_c$$
$$X_1 \sin\phi_1 = X_1^\circ \sin\phi_1^\circ \tag{2.42}$$

so that

$$X_1 e^{i\phi_1} = X_1^\circ e^{i\phi_1^\circ} + X_c \tag{2.43}$$

and

$$X_n e^{i\phi_n} = X_n^\circ e^{i\phi_n^\circ}, \quad n > 1 \tag{2.44}$$

Then Eq. 2.41 becomes

$$-\frac{4}{\gamma+1}\frac{\delta_n}{\cos\theta_n} X_n^\circ e^{i(\phi_n^\circ - \theta_n - \pi/2)} = \sum_{j=1}^{\infty} X_j X_{n+j} e^{i(\phi_{n+j}-\phi_j)} + \frac{1}{2}\sum_{j=1}^{n-1} X_j X_{n-j} e^{i(\phi_{n-j}+\phi_j)} \tag{2.45}$$

If the equivalent pressure equation is desired, make the substitution

$$X = -M \doteq -P/\rho_\circ c_\circ^2 \tag{2.46}$$

where M is the amplitude of the Mach number for the particular harmonic. Define

$$D_n = \beta/2H_n \tag{2.47}$$

so that Eq. 2.52 becomes

$$\frac{M_n^\circ}{D_n} e^{i(\phi_n^\circ - \theta_n - \pi/2)} = \sum_{j=1}^{\infty} M_j M_{n+j} e^{i(\phi_{n+j}-\phi_j)} + \frac{1}{2}\sum_{j=1}^{n-1} M_j M_{n-j} e^{i(\phi_{n-j}+\phi_j)} \tag{2.48}$$

Examination of Eq. 2.48 and Eqs. 2.20 and 2.25 shows that the two methods are equivalent since expansion of the term

$$\frac{1}{2}\sum_{j=1}^{n-1} M_j M_{n-j}\, e^{i(\phi_{n-j}+\phi_j)}$$

yields the leading terms of each harmonic derived in the perturbation scheme, while the other portion of Eq. 2.48 contains the correction terms.

Recall that in both the perturbation approach and the Fourier synthesis approach the leading term of the fundamental was calculated to be $P_{11}\sin K(L-a)\cos\omega t$. This is of order one in the Mach number. It is also a known experimental fact that the fundamental remains essentially unchanged with respect to Mach number. This suggests that the phase angles associated with the corrective terms might cause the corrective terms to sum to approximately zero. It then becomes conceivable that a solution composed of only leading terms could yield significant results. In order to investigate this possibility, Eq. 2.48 was used to obtain the leading terms of the first ten harmonics.

The result is

$$\begin{aligned} p/P_{11} = {} & \frac{1}{2}\cos K(L-a)\Big\{2\sin\omega t\Big\} \qquad (2.49) \\ & + \frac{1}{2}\cos 2K(L-a)\Big\{\Big(\frac{M\beta}{2}\Big)(2)_2\Big\} \\ & + \frac{1}{2}\cos 3K(L-a)\Big\{\Big(\frac{M\beta}{2}\Big)^2(2,3)_3\Big\} \\ & + \frac{1}{2}\cos 4K(L-a)\Big\{\Big(\frac{M\beta}{2}\Big)^3\Big[(2,3,4)_4 \end{aligned}$$

$$+\frac{1}{4}(2,2,4)_4\Big]\Big\}$$

$$+\frac{1}{2}\cos 5K(L-a)\Big\{\left(\frac{M\beta}{2}\right)^4\Big[(2,3,4,5)_5$$

$$+\frac{1}{4}(2,2,4,5)_5+\frac{1}{2}(2,2,3,5)_5\Big]\Big\}$$

$$+\frac{1}{2}\cos 6K(L-a)\Big\{\left(\frac{M\beta}{2}\right)^5\Big[(2,3,4,5,6)_6$$

$$+\frac{1}{4}(2,2,4,5,6)_6+\frac{1}{2}(2,2,3,5,6)_6$$

$$+\frac{1}{2}(2,2,3,4,6)_6+\frac{1}{8}(2,2,2,4,6)_6$$

$$+\frac{1}{4}(2,2,3,3,6)_6\Big]\Big\}$$

$$+\frac{1}{2}\cos 7K(L-a)\Big\{\left(\frac{M\beta}{2}\right)^6\Big[(2,3,4,5,6,7)_7$$

$$+\frac{1}{4}(2,2,4,5,6,7)_7+\frac{1}{2}(2,2,3,5,6,7)_7$$

$$+\frac{1}{2}(2,2,3,4,6,7)_7+\frac{1}{8}(2,2,2,4,6,7)_7$$

$$+\frac{1}{4}(2,2,3,3,6,7)_7+\frac{1}{2}(2,2,3,4,5,7)_7$$

$$+\frac{1}{8}(2,2,2,4,5,7)_7+\frac{1}{4}(2,2,2,3,5,7)_7$$

$$+\frac{1}{2}(2,2,3,3,4,7)_7+\frac{1}{8}(2,2,2,3,4,7)_7\Big]\Big\}$$

$$+\frac{1}{2}\cos 8K(L-a)\Big\{\Big(\frac{M\beta}{2}\Big)^7\Big[(2,3,4,5,6,7,8)_8$$

$$+\frac{1}{4}(2,2,4,5,6,7,8)_8+\frac{1}{2}(2,2,3,5,6,7,8)_8$$

$$+\frac{1}{2}(2,2,3,4,6,7,8)_8+\frac{1}{8}(2,2,2,4,6,7,8)_8$$

$$+\frac{1}{4}(2,2,3,3,6,7,8)_8+\frac{1}{2}(2,2,3,4,5,7,8)_8$$

$$+\frac{1}{8}(2,2,2,4,5,7,8)_8+\frac{1}{4}(2,2,3,5,7,2,8)_8$$

$$+\frac{1}{2}(2,2,3,3,4,7,8)_8+\frac{1}{8}(2,2,2,3,4,7,8)_8$$

$$+\frac{1}{2}(2,2,3,4,5,6,8)_8+\frac{1}{8}(2,2,2,4,5,6,8)_8$$

$$+\frac{1}{4}(2,2,2,3,5,6,8)_8+\frac{1}{4}(2,2,2,3,4,6,8)_8$$

$$+\frac{1}{16}(2,2,2,2,4,6,8)_8+\frac{1}{8}(2,2,2,3,3,6,8)_8$$

$$+\frac{1}{2}(2,2,3,3,4,5,8)_8+\frac{1}{8}(2,2,2,3,4,5,8)_8$$

$$+\frac{1}{4}(2,2,2,3,3,5,8)_8+\frac{1}{4}(2,2,3,3,4,4,8)_8$$

$$+\frac{1}{8}(2,2,2,3,4,4,8)_8+\frac{1}{64}(2,2,2,2,4,4,8)_8\Big]\Big\}$$

$$+\frac{1}{2}\cos 9K(L-a)\Big\{\Big(\frac{M\beta}{2}\Big)^8\Big[(2,3,4,5,6,7,8,9)_9$$

$$+\frac{1}{4}(2,2,4,5,6,7,8,9)_9+\frac{1}{2}(2,2,3,5,6,7,8,9)_9$$

$$+\frac{1}{2}(2,2,3,4,6,7,8,9)_9+\frac{1}{8}(2,2,2,4,6,7,8,9)_9$$

$$+\frac{1}{4}(2,2,3,3,6,7,8,9)_9 + \frac{1}{2}(2,2,3,4,5,7,8,9)_9$$

$$+\frac{1}{8}(2,2,2,4,5,7,8,9)_9 + \frac{1}{4}(2,2,2,3,5,7,8,9)_9$$

$$+\frac{1}{2}(2,2,3,3,4,7,8,9)_9 + \frac{1}{8}(2,2,2,3,4,7,8,9)_9$$

$$+\frac{1}{2}(2,2,3,4,5,6,8,9)_9 + \frac{1}{8}(2,2,2,4,5,6,8,9)_9$$

$$+\frac{1}{4}(2,2,2,3,5,6,8,9)_9 + \frac{1}{4}(2,2,2,3,4,6,8,9)_9$$

$$+\frac{1}{16}(2,2,2,2,4,6,8,9)_9 + \frac{1}{8}(2,2,2,3,3,6,8,9)_9$$

$$+\frac{1}{2}(2,2,3,3,4,5,8,9)_9 + \frac{1}{8}(2,2,2,3,4,5,8,9)_9$$

$$+\frac{1}{4}(2,2,2,3,3,5,8,9)_9 + \frac{1}{4}(2,2,3,3,4,4,8,9)_9$$

$$+\frac{1}{8}(2,2,2,3,4,4,8,9)_9 + \frac{1}{64}(2,2,2,2,4,4,8,9)_9$$

$$+\frac{1}{2}(2,2,3,4,5,6,7,9)_9 + \frac{1}{8}(2,2,2,4,5,6,7,9)_9$$

$$+\frac{1}{4}(2,2,2,3,5,6,7,9)_9 + \frac{1}{4}(2,2,2,3,4,6,7,9)_9$$

$$+\frac{1}{16}(2,2,2,2,4,6,7,9)_9 + \frac{1}{8}(2,2,2,3,3,6,7,9)_9$$

$$+\frac{1}{4}(2,2,2,3,4,5,7,9)_9 + \frac{1}{16}(2,2,2,2,4,5,7,9)_9$$

$$+\frac{1}{8}(2,2,2,2,3,5,7,9)_9 + \frac{1}{4}(2,2,2,3,3,4,7,9)_9$$

$$+\frac{1}{16}(2,2,2,2,3,4,7,9)_9 + \frac{1}{2}(2,2,3,3,4,5,6,9)_9$$

$$+\frac{1}{8}(2,2,2,3,4,5,6,9)_9 + \frac{1}{4}(2,2,2,3,3,5,6,9)_9$$

$$+\frac{1}{4}(2,2,2,3,3,4,6,9)_9 + \frac{1}{16}(2,2,2,2,3,4,6,9)_9$$

$$+\frac{1}{8}(2,2,2,3,3,3,6,9)_9 + \frac{1}{2}(2,2,3,3,4,4,5,9)_9$$

$$+\frac{1}{8}(2,2,2,3,4,4,5,9)_9 + \frac{1}{4}(2,2,2,3,3,4,5,9)_9$$

$$+\frac{1}{8}(2,2,2,3,4,4,5,9)_9 + \frac{1}{32}(2,2,2,2,4,4,5,9)_9$$

$$+\frac{1}{16}(2,2,2,2,3,4,5,9)_9\Big]\Big\}$$

$$+\frac{1}{2}\,\mathrm{Cos}\,10K(L-a)\left\{\left(\frac{M\beta}{2}\right)^9\Big[(2,3,4,5,6,7,8,9,10)_{10}$$

$$+\frac{1}{4}(2,2,4,5,6,7,8,9,10)_{10} + \frac{1}{2}(2,2,3,5,6,7,8,9,10)_{10}$$

$$+\frac{1}{2}(2,2,3,4,6,7,8,9,10)_{10} + \frac{1}{8}(2,2,2,4,6,7,8,9,10)_{10}$$

$$+\frac{1}{4}(2,2,3,3,6,7,8,9,10)_{10} + \frac{1}{2}(2,2,3,4,5,7,8,9,10)_{10}$$

$$+\frac{1}{8}(2,2,2,4,5,7,8,9,10)_{10} + \frac{1}{4}(2,2,2,3,5,7,8,9,10)_{10}$$

$$+\frac{1}{2}(2,2,3,3,4,7,8,9,10)_{10} + \frac{1}{8}(2,2,2,3,4,7,8,9,10)_{10}$$

$$+\frac{1}{2}(2,2,3,4,5,6,8,9,10)_{10} + \frac{1}{8}(2,2,2,4,5,6,8,9,10)_{10}$$

$$+\frac{1}{4}\;2,2,2,3,5,6,8,9,10)_{10} + \frac{1}{4}(2,2,2,3,4,6,8,9,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,4,6,8,9,10)_{10}+\frac{1}{8}(2,2,2,3,3,6,8,9,10)_{10}$$

$$+\frac{1}{2}(2,2,3,3,4,5,8,9,10)_{10}+\frac{1}{8}(2,2,2,3,4,5,8,9,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,3,5,8,9,10)_{10}+\frac{1}{4}(2,2,3,3,4,4,8,9,10)_{10}$$

$$+\frac{1}{16}(2,2,2,3,4,4,8,9,10)_{10}+\frac{1}{64}(2,2,2,2,4,4,8,9,10)_{10}$$

$$+\frac{1}{2}(2,2,3,4,5,6,7,9,10)_{10}+\frac{1}{8}(2,2,2,4,5,6,7,9,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,5,6,7,9,10)_{10}+\frac{1}{4}(2,2,2,3,4,6,7,9,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,4,6,7,9,10)_{10}+\frac{1}{8}(2,2,2,3,3,6,7,9,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,4,5,7,9,10)_{10}+\frac{1}{16}(2,2,2,2,4,5,7,9,10)_{10}$$

$$+\frac{1}{8}(2,2,2,2,3,5,7,9,10)_{10}+\frac{1}{4}(2,2,2,3,3,4,7,9,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,3,4,7,9,10)_{10}+\frac{1}{2}(2,2,3,3,4,5,6,9,10)_{10}$$

$$+\frac{1}{8}(2,2,2,3,4,5,6,9,10)_{10}+\frac{1}{4}(2,2,2,3,3,5,6,9,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,3,4,6,9,10)_{10}+\frac{1}{16}(2,2,2,2,3,4,6,9,10)_{10}$$

$$+\frac{1}{8}(2,2,2,3,3,3,6,9,10)_{10}+\frac{1}{2}(2,2,3,3,4,4,5,9,10)_{10}$$

$$+\frac{1}{8}(2,2,2,3,4,4,5,9,10)_{10}+\frac{1}{4}(2,2,2,3,3,4,5,9,10)_{10}$$

$$+\frac{1}{8}(2,2,2,3,4,4,5,9,10)_{10}+\frac{1}{32}(2,2,2,2,4,4,5,9,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,3,4,5,9,10)_{10}+\frac{1}{2}(2,2,3,4,5,6,7,8,10)_{10}$$

$$+\frac{1}{8}(2,2,2,4,5,6,7,8,10)_{10} + \frac{1}{4}(2,2,2,3,5,6,7,8,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,4,6,7,8,10)_{10} + \frac{1}{16}(2,2,2,2,4,6,7,8,10)_{10}$$

$$+\frac{1}{8}(2,2,2,3,3,6,7,8,10)_{10} + \frac{1}{4}(2,2,2,3,4,5,7,8,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,4,5,7,8,10)_{10} + \frac{1}{8}(2,2,2,2,3,5,7,8,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,3,4,7,8,10)_{10} + \frac{1}{16}(2,2,2,2,3,4,7,8,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,4,5,6,8,10)_{10} + \frac{1}{16}(2,2,2,2,4,5,6,8,10)_{10}$$

$$+\frac{1}{8}(2,2,2,2,3,5,6,8,10)_{10} + \frac{1}{8}(2,2,2,2,3,4,6,8,10)_{10}$$

$$+\frac{1}{32}(2,2,2,2,2,4,6,8,10)_{10} + \frac{1}{16}(2,2,2,2,3,3,6,8,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,3,4,5,8,10)_{10} + \frac{1}{16}(2,2,2,2,3,4,5,8,10)_{10}$$

$$+\frac{1}{8}(2,2,2,2,3,3,5,8,10)_{10} + \frac{1}{8}(2,2,2,3,3,4,4,8,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,3,4,4,8,10)_{10} + \frac{1}{128}(2,2,2,2,2,4,4,8,10)_{10}$$

$$+\frac{1}{2}(2,2,3,3,4,5,6,7,10)_{10} + \frac{1}{8}(2,2,2,3,4,5,6,7,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,3,5,6,7,10)_{10} + \frac{1}{4}(2,2,2,3,3,4,6,7,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,3,4,6,7,10)_{10} + \frac{1}{8}(2,2,2,3,3,3,6,7,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,3,4,5,7,10)_{10} + \frac{1}{16}(2,2,2,2,3,4,5,7,10)_{10}$$

$$+\frac{1}{8}(2,2,2,2,3,3,5,7,10)_{10} + \frac{1}{4}(2,2,2,3,3,3,4,7,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,3,3,4,7,10)_{10}+\frac{1}{2}(2,2,3,3,4,4,5,6,10)_{10}$$

$$+\frac{1}{8}(2,2,2,3,4,4,5,6,10)_{10}+\frac{1}{4}(2,2,2,3,3,4,5,6,10)_{10}$$

$$+\frac{1}{4}(2,2,2,3,3,4,4,6,10)_{10}+\frac{1}{16}(2,2,2,2,3,4,4,6,10)_{10}$$

$$+\frac{1}{8}(2,2,2,3,3,3,4,6,10)_{10}+\frac{1}{8}(2,2,2,3,4,4,5,6,10)_{10}$$

$$+\frac{1}{32}(2,2,2,2,4,4,5,6,10)_{10}+\frac{1}{16}(2,2,2,2,3,4,5,6,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,3,4,4,6,10)_{10}+\frac{1}{64}(2,2,2,2,2,4,4,6,10)_{10}$$

$$+\frac{1}{32}(2,2,2,2,3,3,4,6,10)_{10}+\frac{1}{4}(2,2,3,3,4,4,5,5,10)_{10}$$

$$+\frac{1}{64}(2,2,2,2,4,4,5,5,10)_{10}+\frac{1}{16}(2,2,2,2,3,3,5,5,10)_{10}$$

$$+\frac{1}{8}(2,2,2,3,4,4,5,5,10)_{10}+\frac{1}{4}(2,2,2,3,3,4,5,5,10)_{10}$$

$$+\frac{1}{16}(2,2,2,2,3,4,5,5,10)_{10}\Bigg]\Bigg\}$$

$$+\cdots$$

where (a,b)g has been given in Eq. 2.21.

Cavity Resonance Response

Investigation of the behavior near resonance of each approach leads to the desirability of obtaining the response of the cavity for each harmonic as the fundamental is swept through resonance. It is obvious from the form of Eqs. 2.27, 2.32, and 2.55 that both approaches yield results which may be written as sums of terms having like factors of K(L-a) and ωt. It is also evident that each of these harmonics is heavily dependent upon M, the Mach number, and $\Delta\omega$.

The algebraic process necessary to bring each of the two solutions to a form which conveniently yields the desired information will be derived in a general form and then applied to the perturbation solution. It is then easily extended to the Fourier synthesis solution.

We begin with the relationship

$$\sum_{i=1}^{n} a_i \sin(j\omega t + \phi_i) = A \sin(j\omega t + \phi). \qquad (2.50)$$

At t = 0 Eq. 2.50 becomes

$$\sum_{i=1}^{n} a_i \sin \phi_i = A \sin \phi. \qquad (2.51)$$

The derivative of Eq. 2.50 evaluated at t = 0 yields

$$\sum_{i=1}^{n} a_i \cos \phi_i = A \cos \phi \qquad (2.52)$$

which when combined with Eq. 2.51, yields

$$A^2 = \left(\sum_{i=1}^{n} a_i \sin \phi_i\right)^2 + \left(\sum_{i=1}^{n} a_i \cos \phi_i\right)^2. \qquad (2.53)$$

If Eq. 2.51 is divided by Eq. 2.52 the result is

$$\tan \phi = \sum_{i=1}^{n} a_i \sin \phi_i \Big/ \sum_{i=1}^{n} a_i \cos \phi_i \qquad (2.54)$$

The sum of Eqs. 2.20 and 2.25 may be written as

$$p_i/p_{11} = \frac{1}{2}\cos jK(L-a)\sum_{i=1}^{n} a_i \sin(j\omega t+\phi_i) \qquad (2.55)$$
$$= \frac{A}{2}\cos jK(L-a)\sin j\omega t+\phi.$$

for each harmonic, where j indicates the harmonic number and n indicates the number of terms in that harmonic. It is evident that the amplitude of each harmonic may be obtained by applying Eq. 2.53.

In applying Eq. 2.53 to the perturbation approach the magnitude of each harmonic was normalized to approximately one by the maximum magnitude of a factor chosen from the leading term of that harmonic. This normalization was done to yield a set of curves whose maxima are of order unity. For the fundamental this normalization factor is one. The method of choosing the normalization factor for higher harmonics will be demonstrated for the second harmonic. The leading term of the second harmonic is

$$\left(\frac{M\beta}{2}\right)(2)_2 = \frac{1}{2}\frac{M\beta}{2}\frac{1}{H_2}\sin(2\omega t+\theta_2). \qquad (2.56)$$

Now

$$M = \frac{U_{11}}{C_0} = \frac{2A_0}{m\pi\omega_\mu C_0 H_1} \qquad (2.57)$$

so that

$$\frac{M\beta}{2} = \frac{A_0\beta}{m\pi\omega_\mu C_0 H_1} \qquad (2.58)$$

and the leading term of the second harmonic may be written

$$\frac{A_0\beta}{2m\pi\omega_\mu C_0}\,\frac{1}{H_1 H_2}\sin(2\omega t+\theta_2) \qquad (2.59)$$

Since

$$H_n = \delta_n / \cos\theta_n \tag{2.60}$$

Eq. 2.59 may be written

$$\frac{A_o\beta}{2m\pi\omega_p C_o} \quad \frac{\cos\theta_1}{\delta_1} \quad \frac{\cos\theta_2}{\delta_2} \quad \sin(2\omega t + \theta_2) \tag{2.61}$$

H_n has as its maximum

$$[H_n]_{max} = \delta_n \tag{2.62}$$

and the normalization factor for the second harmonic is then chosen as

$$\frac{A_o\beta}{2m\pi\omega_p C_o} \quad \frac{1}{\delta_1\delta_2} \tag{2.63}$$

It should be noted that these normalization factors are not the maximum amplitudes of the various harmonics, neither are they the maximum amplitudes of the leading terms of each harmonic; rather, they are factors chosen to yield maxima of order unity.

These factors are given the symbol $MAXA_n$ where n refers to the n-th harmonic so that

$$MAXA_2 = \frac{A_c\beta}{2m\pi\omega_p C_o} \quad \frac{1}{\delta_1\delta_2} \tag{2.64}$$

By combining Eqs. 2.53 and 2.64 it is possible to define Q_n as the normalized response of the n-th harmonic. It should be noted that A defined in Eq. 2.53 is a function of the Mach number and $\Delta\omega$.

Thus

$$Q_n(M, \Delta\omega) = A_n / MAXA_n \tag{2.65}$$

where n is the harmonic number.

3. APPLICATIONS

Associated with each of the solutions formulated in the previous section are three computer programs. These programs are concerned with the pressure waveform, the Q-curves, and the phase information. The perturbation solution has the programs FINAMPI, QCURVES, and PHAMP connected with it while FINAMPIZ, QCURC and PHAMPC are related to the Fourier synthesis.

The first attempted utilization of the perturbation solution consisted of writing a computer program involving all terms through sixth order. This program, entitled FINAMPI, was designed to compute and graph both velocity and pressure waveforms. FINAMPI was first used to graph the waveforms contained in Ref. [4] and has been used since in determining for what Mach numbers the theory begins to deviate significantly from the experimental results. The graphical output from the computer is contained in Figs. C.1 through C.4.

The program designed to compute and graph the cavity resonance response for the perturbation solution is entitled QCURVES. This program is a direct application of Eq. 2.65 and has been used to obtain the curves contained in Figs. C.5 through C.22.

PHAMP, the third program associated with the perturbation solution, was designed to compute the phase angles of each harmonic as the fundamental was swept through resonance. It is a direct application of Eq. 2.54 and resulted in the data contained in Table 4.2.

In an attempt both to determine the importance of the corrective factors to each harmonic and to test the validity of the Fourier synthesis, computer programs involving only the leading terms of the various harmonics were written. Unfortunately a program involving all ten harmonics could not be brought to fruition because of time limitations. A second program, FINAMPIZ, which uses the leading terms of the first six harmonics has been used to graph the pressure waveforms also contained in Figs. C.1 through C.4.

Eqs. 2.65 and 2.54 were also applied to the Fourier synthesis yielding the programs QCURC and PHAMPC. QCURC computes and graphs the Q-curves for each of the harmonics while PHAMPC computes and graphs the phase dependence of the individual harmonics on $\Delta\omega$.

It should be noted that of the two programs concerned with phase dependence, only PHAMPC graphs its output. This was done because the Q-curve programs indicated the Fourier synthesis approach produced curves which were in much better agreement with the experimental results than did the perturbation approach. The graphical output of PHAMPC is contained in Fig. C.23.

In this section only a superficial description of each program was given. A detailed description of all programs is contained in Appendix B, and a more complete discussion of results follows in Section 4.

4. RESULTS AND COMPARISONS

In this section the theoretical predictions obtained from the applications described in Section 3 will be compared to the experimental results obtained by Beech [19].

Before proceeding to a discussion of these results it is necessary to attempt some qualitative evaluation of the apparent properties of the perturbation solution. The amplitudes of the (n + 1)th order terms tend to behave as [4].

$$n\left(\frac{M\beta}{2}\right)^n \prod_{j=2}^{n} \frac{1}{H_j} = n\left(\frac{M\beta}{2}\right)^n \sqrt{n!} \prod_{j=2}^{n} \cos\theta_j \tag{4.1}$$

It is clear that this expression can diverge for increasing n. This divergence, which occurs because of the dependence $\delta_n = \delta_1/\sqrt{n}$, would not occur if $\delta_n \geq \delta_1$ since the left hand side of Eq. 4.1 would then be bounded by $n\left(M\beta/2\delta_1\right)^n$ and all series would converge for $M\beta < 2\delta_1$. The Mach number which satisfies the expression $M\beta/2\delta_1 = 1$ and would therefore appear to be related to shock formation is M = 0.02. This is within the same order of magnitude as the experimentally measured Mach number at shock formation, M = 0.01.

The divergence may be remedied if it is recalled that the bulk absorptive processes have been neglected up to this point. If the viscous terms are retained and assumed to be additive to the wall losses then

$$D_j \doteq \left[\frac{-\delta_j}{\omega_j} + \epsilon\right]\frac{\partial^3}{\partial a^2 \partial t} - \delta_j \frac{\partial^2}{\partial a^2} \tag{4.2}$$

where

$$\epsilon = \frac{1}{\rho_0 c_0^2}\left[\frac{4}{3}\eta + \eta_B\right]. \tag{4.3}$$

The analysis and results are the same as before with the H's and Θ's redefined as

$$H_n = \frac{\delta_n + n\omega\epsilon}{\cos\theta_n} \tag{4.4}$$

$$\theta_n = \tan^{-1}\frac{\delta_1 - \delta_n - 2\Delta\omega/\omega_\mu}{\delta_n + n\omega\epsilon}\,. \tag{4.5}$$

With these modifications the amplitudes of the (n + 1)th order terms now behave, for sufficiently large harmonic, as

$$n\left(\frac{M\beta}{2}\right)^n \prod_{j=1}^{n} \frac{1}{H_j} = n\left(\frac{M\beta}{4\omega\epsilon}\right)^n (n!)^{-1} \tag{4.6}$$

which tends to zero with increasing n.

The modifications introduced by Eqs. 4.2 and 4.3 are important in the perturbation expansion for those harmonics whose indices satisfy the inequality $n^{3/2} \gtrsim \delta_1/\omega\epsilon$. For the problem dealt with in this research the theory developed without the bulk absorptive terms is adequate as long as attention is restricted to frequencies below about the thousandth overtone.

In comparing the theoretical and the experimental results it is convenient to divide the discussion into three areas: the low Mach number region, the intermediate Mach number region and the high Mach number region.

In the low Mach number region, below $M \approx 0.006$, it is obvious from Figs. C.1 and C.2 that theory and experiment are in very close agreement. In these figures of pressure waveforms the large drawing contains the waveforms predicted from the computer programs and the smaller inset is a drawing from an oscilloscope photograph of the waveform observed experimentally. The dashed line in both drawings is a sine wave which is included for reference purposes. One of the main reasons for the good agreement in this region is due to the fact that the $(M\beta/2\delta_1)$ terms are as yet relatively small.

It should be noted here that in all the figures of pressure waveforms, the major difference between the FINAMPI and FINAMPIZ predictions is in the phase dependence. Recall that the development of the reasoning behind the FINAMPIZ program rested on the possibility that corrective terms to the generated harmonics could be ignored. These curves lend support to that suggestion in the low Mach number region.

The low Mach number region being discussed also includes the first two sets of Q-curves, Figs. C.5 through C.16. It is evident in both sets that the agreement of the experimental results with the predictions of the Fourier synthesis approach is excellent in the lower harmonics and deviates only slightly in the higher harmonics. When the agreement does deviate, the experimental results in general fall between the two theoretical predictions. This tends to indicate that the two solutions have the exact solution bracketed. It is evident in the QCURVES predictions for M = 0.005 that the third harmonic curve is no longer a smooth bell-shaped curve. The reason for this is the growth in the third harmonic of the correction terms arising from the fifth-order solution. For instance, the ratio of the approximate magnitude of the first corrective term to the leading term in the third harmonic is about 0.5 at M = 0.005. This is no longer insignificant so that the predictions based on the perturbation approach begin to show unrealistic fluctuations.

Theoretically predicted amplitude information is obtained from the two programs, PHAMP and PHAMPC. The amplitude results from these programs together with the experimental results are contained in Table 4.1. It should be noted that Beech's uncertainty of ± 0.1 Hz in frequency results in an uncertainty of ± 0.628 radians in Figs. C.5 through C.23. A comparison of the uncertainty in amplitude (based on the uncertainty in $\Delta\omega$) with the observed discrepancy between measurement and theory for $M = 0.004$ and $\Delta\omega = 0$ yields

n	uncertainty in harmonic content	discrepancy between PHAMPC and experiment
2	0.5	0.4
3	0.3	0.17
4	0.15	0.12
5	0.07	0.08
6	0.03	0.07

It is evident that the uncertainties in harmonic content are consistent with the discrepancies in harmonic content noted between PHAMPC and experiment.

As was stated in Section 3, PHAMPC also predicts the phase dependence of the various harmonics. The results of this program are contained in Table 4.2 and Fig. C.23. One thing should be noted. The sets of phase angles predicted by PHAMPC are identical for all Mach numbers, which is not surprising if the reader recalls that the normalization process destroyed the Mach-number dependence of the leading terms of each harmonic.

Program	Harmonic Number	Mach Number			Program	Harmonic Number	Mach Number		
		0.004	0.005	0.009			0.004	0.005	0.009
		Percent Harmonic Content					Percent Harmonic Content		
PHAMP	2	10.9	13.6	20.2	FOUANAL	2	12.0	17.5	23.2
	3	2.78	4.28	14.6		3	3.5	6.9	7.8
	4	1.06	1.83	8.55		4	1.73	3.31	4.12
	5	0.567	1.37	13.7		5	1.97	2.35	2.85
	6	0.27	0.815	13.9		6	0.93	1.03	1.95
PHAMPC	2	12.2	15.2	27.4	Experimental Results [19]	2	11.8	17.0	21.1
	3	3.47	5.4	17.5		3	3.3	5.8	9.6
	4	1.32	2.57	15.0		4	1.2	2.4	5.6
	5	0.6	1.4	14.73		5	0.52	1.1	3.8
	6	0.272	0.84	15.9		6	0.20	0.57	2.5

Tabulated Theoretical and Experimental Values of the Harmonic Content in the Waveforms, $\Delta\omega = 0$

Table 4.1

One other program exists for the investigation of phase and amplitude characteristics. The computer program FOUANAL is being introduced here since it was not derived from either theoretical approach. FOUANAL takes data from an oscilloscope photograph and computes the Fourier coefficients of the various terms of a Fourier coefficients of the various terms of a Fourier expansion of the waveform. The results from this program are also contained in Tables 4.1 and 4.2.

It is clear that the results from FOUANAL are fairly accurate in the lower harmonics (first, second and third) but tend to become much worse in the higher harmonics (fourth, fifth and sixth). Judging from the amplitude results it does not seem likely that FOUANAL would yield accurate results in the phase angles of these higher harmonics. Examination of Table 4.2 verifies this supposition. These erratic results in the FOUANAL calculations may possibly be explained in the following manner. The input data for the program come from extracting sixty-four points from a standard oscilloscope photograph. This was done by hand and was subject to many errors. It is possible that the data were not given to sufficient accuracy to enable the computer to predict accurately past the third harmonic. The most feasible suggestion for improvement in this respect is to use a digital counting procedure to obtain the data. It should be noted that the data in Fig. C.23 are the FOUANAL results for M = 0.004. Clearly these points are extremely random and the uncertainty in the experimental $\Delta\omega$ cannot possibly account for the entire discrepancy.

Shifting now to the medium Mach number region ($0.006 < M < 0.009$) we see from Fig. C.3, for M = 0.007, that although the general agreement is still good, deviations from the previous good agreement are becoming

Program	Harmonic Number	Mach Number		
		0.004	0.005	0.009
		Phase Angle (Radians)		
FOUANAL	1	0.0	0.0	0.0
	2	0.1	0.178	0.103
	3	0.58	0.54	0.446
	4	0.76	1.15	0.778
	5	0.93	1.71	1.145
	6	1.33	2.27	1.04
PHAMPC	1	0.0	0.0	0.0
	2	0.1866	0.1866	0.1866
	3	0.431	0.431	0.431
	4	0.603	0.603	0.603
	5	0.851	0.851	0.851
	6	1.570	1.570	1.570

Tabulated Theoretical Values of the Harmonic Phase Angles

Table 4.2

evident in the computer predictions. Unfortunately experimental data for Q-curve and phase angle comparisons were not obtained but it is not unreasonable to assume that these comparisons would show essentially the same result as stated for the pressure waveform.

Moving into the region of high Mach numbers, M = 0.009, we find that the pressure waveforms predicted by each program are generally invalid. This is nearing the experimentally observed Mach number for shock formation (M = 0.01), and since the theory is predicted on small Mach number, pre-shock conditions it is not surprising to see these deviations.

Essentially the same results are observed in the Q-curve comparisons, Figs. C.17 through C.22. The reason for the experimental amplitudes falling off so rapidly is at present unknown. Tables 4.1 and 4.2 again reflect the serious discrepancies in the high Mach number region.

It is now possible to make some general comments concerning these two approaches. Both the perturbation approach and the Fourier synthesis approach have adequately described the effects of finite-amplitude standing waves in rigid-walled cavities provided the Mach number is relatively low. This precludes that area in the pre-shock region where significant distortion exists. It has been demonstrated that in the region where the approaches are fairly accurate, Q-curve information can be easily obtained. This information is convenient for comparisons with experimental results. It is also possible to obtain some information on the phase angles of the harmonics.

It seems plausible now to suggest the following: Since the Fourier synthesis approach gave the most accurate information, the computer program involving all ten harmonics should be reviewed in the hope that a usable program will result. This program could be used to investigate the effects of the higher order harmonics on the existing solution and possibly give an indication of whether extending the present developments past six harmonics is beneficial.

Finally, after some exposure to basic communications theory, it appears that a good grounding in this subject is mandatory if the present theoretical line is to be pursued further. It would not be surprising to find that a more exact solution may be obtained as a result of the techniques used in communications theory.

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APPENDIX A

Coppens-Sanders Iteration Scheme

From Section 2, the approximate wave equation for finite-amplitude standing waves in a rigid walled tube is

$$\sum_j{}' \left(\Box^2 + \frac{\delta_j}{\omega_j} \frac{\partial^3}{\partial a^2 \partial t} - \delta_j \frac{\partial^2}{\partial a^2} \right) u_{nj} = \beta \frac{\partial^2}{\partial a \partial t} \sum_{i=1}^{n-1} \frac{\partial \xi_i}{\partial a} \frac{\partial \xi_{n-i}}{\partial a} = \sum_j \phi_{nj} \tag{A.1}$$

For convenience the right hand side of Eq. A.1 will be written

$$\phi_n = \sum_j \phi_{nj} = \beta \frac{\partial^2}{\partial a \partial t} \sum_j{}' \psi_{nj} \tag{A.2}$$

Assume

$$\underset{\sim}{u}_{nj} = \underset{\sim}{U}_{nj} \sin jK(L-a) \, e^{i(j\omega t + \cdots)} \tag{A.3}$$

where the $\underset{\sim}{u}_{nj}$ and $\underset{\sim}{U}_{nj}$ are complex, and the three dots in the exponential term represent phase terms to be defined. Work is done in one n at a time so that the n subscript may be dropped.

Expansion of the left hand side of Eq. A.1 yields

$$\frac{\partial^2}{\partial a^2} - \delta_j \frac{\partial^2}{\partial a^2} = (1-\delta_j) \frac{\partial^2}{\partial a^2} \tag{A.4}$$

where $(1-\delta_j)$ was previously defined to be $(c_j/c_0)^2$.

Thus Eq. A.1 becomes

$$\left(\frac{c_j}{c_0}\right)^2 \frac{\partial^2}{\partial a^2} u_j - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} u_j + \frac{\delta_j}{\omega_j} \frac{\partial^3 u_j}{\partial a^2 \partial t} = \phi_j \tag{A.5}$$

for a given order n. If Eq. A.5 is generalized in time it becomes

$$\left(\frac{c_j}{c_0}\right)^2 \frac{\partial^2 \underset{\sim}{u}_j}{\partial a^2} - \frac{1}{c_0^2} \frac{\partial^2 \underset{\sim}{u}_j}{\partial t^2} + \frac{\delta_j}{\omega_j} \frac{\partial^3 \underset{\sim}{u}_j}{\partial a^2 \partial t} = \not{\mathcal{L}}_j \tag{A.6}$$

Substitution of Eq. A.3 into Eq. A.6 yields

$$\left(-c_j^2 j^2 K^2 + j^2\omega^2 - i c_o^2 \delta_j j^2 K^2\right)\underset{\sim}{u}_j = \underset{\sim}{\phi}_{nj}\, c_o^2 \tag{A.7}$$

Now each ψ will be of the form

$$A_{nj} \cos jK(L-a) \cos (j\omega t + \cdots) \tag{A.8}$$

Substitution yields

$$\begin{aligned}\phi_{nj} &= -A_{nj}\beta j K\left[\sin jK(L-a)\right] j\omega \sin(j\omega t+\cdots)\\ &= A_{nj}\beta_j K j\omega\left[\sin jK(L-a)\right] i e^{i(j\omega t+\cdots)}\end{aligned} \tag{A.9}$$

Additional substitution of Eq. A.9 and Eq. A.3 into Eq. A.7 gives

$$\underset{\sim}{U}_{nj}\left[-\left(\frac{c_j}{c_o}\right)^2 + \frac{\omega^2}{c_o^2 K^2} - i\delta_j\right] = i A_{nj}\beta\frac{\omega}{K} \tag{A.10}$$

Proper manipulation of the bracketed term in Eq. A.10 will yield the phase relationships mentioned previously but left undefined.

First recall that

$$\begin{aligned}\omega &= \omega_p + \Delta\omega\\ \frac{\omega_p}{c} &= \frac{m\pi}{L} - \frac{\alpha^2 L}{m\pi} = K - \frac{\alpha^2}{K}\end{aligned} \tag{A.11}$$

where

$$K = m\pi/L\,,$$

$$(c_j/c_o)^2 = 1-\delta_j$$

and notice that

$$\frac{\omega}{K} = \frac{\omega_p+\Delta\omega}{K} = c\left(1-\frac{\alpha^2}{K^2}\right) + \frac{\Delta\omega}{K}\,, \tag{A.12}$$

and

$$\alpha \doteq \frac{\omega}{c_o}\,\frac{1}{2}\,\delta_1\left(1+\frac{3}{2}\delta_1\right), \tag{A.13}$$

$$c \doteq c_0 \left(1 + \frac{1}{2}\delta_1 + \frac{1}{2}\delta_1^{2}\right)^{-1} \tag{A.14}$$

Substitution into Eq. A.10 yields

$$\underset{\sim}{U}_{nj} \left(\delta_j - \delta_1 + \frac{2\Delta\omega}{\omega_\mu} - i\delta_j\right) = i A_{nj} \beta \frac{\omega}{K} \tag{A.15}$$

It is now possible to write down the now U as

$$\underset{\sim}{U}_{nj} = \frac{i A_{nj}\, \beta\, \omega/K}{\delta_j - \delta_1 + \frac{2\Delta\omega}{\omega_\mu} - i\delta_j} \tag{A.16}$$

Define the following quantities;

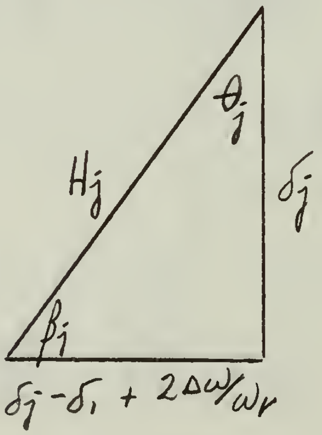
$$H_j \equiv \sqrt{\left(\delta_j - \delta_1 + \frac{2\Delta\omega}{\omega_\mu}\right)^2 + \delta_j^{2}} \tag{A.17}$$

$$\tan \beta_j = \frac{\delta_j}{\delta_j - \delta_1 + 2\Delta\omega/\omega_\mu} \tag{A.18}$$

and

$$\theta_j = -\cot^{-1} \frac{\delta_j}{\delta_j - \delta_1 + 2\Delta\omega/\omega_\mu} \tag{A.19}$$

Pictorially these definitions may be represented as in Fig. A.1.



Pictorial representation of the quantities
defined in Eqs. A.17 through A.19.

Figure A.1

With these definitions, Eq. A.16 becomes

$$\begin{aligned}\underset{\sim}{u}_{nj} &= -A_{nj}\frac{1}{H_j}\beta\frac{\omega}{K}e^{i\theta_j}\\ &= -A_{nj}\frac{1}{H_j}\beta c_0 e^{i\theta_j}\end{aligned} \tag{A.20}$$

and

$$\psi_{nj} = A_{nj}\cos K(L-a)\cos(j\omega t+\dots) = \left[\frac{\partial\xi_i}{\partial a}\frac{\partial\xi_{n-i}}{\partial a}\right]_j \tag{A.21}$$

yielding

$$\underset{\sim}{u}_{nj} = \underset{\sim}{\bar{U}}_{nj}\sin jK(L-a)e^{i(j\omega t+\dots)} \tag{A.22}$$

The preceding was a general outline of the procedure to be followed in the Coppens-Sanders iteration scheme. The remainder of this section contains a derivation of the terms through third order.

Assume the input

$$u_1 = \bar{U}_{11}\sin K(L-a)\cos\omega t \tag{A.23}$$

which was shown in Section 2 to be the calculated first-order solution. Then we have

$$\xi_1 = \frac{\bar{U}_{11}}{\omega}\sin K(L-a)\sin\omega t \tag{A.24}$$

$$\frac{\partial\xi_1}{\partial a} = -K\frac{\bar{U}_{11}}{\omega}\cos K(L-a)\sin\omega t \tag{A.25}$$

and since $\omega/K \doteq c_0$,

$$\frac{\partial\xi_1}{\partial a} = -\frac{\bar{U}_{11}}{c_0}\cos K(L-a)\sin\omega t. \tag{A.26}$$

The next step is to obtain ξ_2.

$$\psi_2 = \sum_{i=1}^{1}\frac{\partial\xi_i}{\partial a}\frac{\partial\xi_{n-i}}{\partial a} = \frac{\partial\xi_1}{\partial a}\frac{\partial\xi_1}{\partial a} \tag{A.27}$$

Substitution of Eq. A.26 into Eq. A.27 yields

$$\psi_2 = M^2 \cos^2 K(L-a) \sin^2 \omega t \tag{A.28}$$

where M = $\bar{U}_{11}/c_0$

Expansion of the trigonometric term gives

$$\cos^2 K(L-a) \sin^2 \omega t \tag{A.29}$$

$$= \left(-\frac{1}{4}\right)\left[\cos\left\{2K(L-a)\right\}+1\right]\left[\cos(2\omega t)-1\right]$$

from which the constants + 1 and - 1 may be discarded since application of $\partial^2/\partial a \partial t$ will remove them. Then

$$\psi_2 = M^2\left(-\frac{1}{4}\right) \cos 2K(L-a) \cos 2\omega t \tag{A.30}$$

which implies

$$A_{22} = -\frac{1}{4} M^2 \tag{A.31}$$

and from Eq. A.20

$$\bar{U}_{22} = \frac{1}{4} M^2 \frac{1}{H_2} \beta c_0 e^{i\theta_2}$$

$$= \frac{1}{2} \bar{U}_{11} \left(\frac{M\beta}{2}\right) \frac{1}{H_2} e^{i\theta_2} \tag{A.32}$$

Recall that $u_{nj} = \mathcal{R}_e\, \underset{\sim}{u}_{nj}$, and hence,

$$u_2 = \frac{1}{2} \bar{U}_{11} \left(\frac{M\beta}{2}\right) \frac{1}{H_2} \sin 2K(L-a) \cos(2\omega t + \theta_2) \tag{A.33}$$

which is the second order solution.

The general equation for $\partial\xi/\partial a$ is

$$\frac{\partial \xi_m}{\partial a} = -\frac{U}{c_0} \cos mK(L-a) \sin(m\omega t + \cdots). \tag{A.34}$$

Substitution of Eq. A.34 into Eq. A.21 results in trigonometric products such as

$$\cos mK(L-a) \sin(m\omega t + \gamma_m) \cos lK(L-a) \sin(l\omega t + \gamma_l) \tag{A.35}$$

For simplicity let

$$X = K(L-a)$$
$$T = \omega t \qquad \text{(A.36)}$$

Then Eq. A.35 may be written

$$\cos mX \sin(mT+\delta_m) \cos lX \sin(lT+\delta_l) \qquad \text{(A.37)}$$

Expansion of Eq. A.37 gives

$$\left(-\tfrac{1}{4}\right)\Big[\cos(m+l)X \cos\{(m+l)T+\delta_m+\gamma_l\}$$
$$- \cos(m-l)X \cos\{(m-l)T+\delta_m-\delta_l\} \qquad \text{(A.38)}$$

where terms of the form sin $mK(L-a)$ cos $l\omega t$ have been ignored in Eq. A.38 since substitution of these terms into the perturbed wave equation result in amplitudes of order α/k which have been neglected throughout. Eq. A.38 must always be written as $(m - l)$ where $m > l$, since a positive frequency has always been assumed.

Before proceding to the derivation of the third harmonic terms, a brief discussion might be of some use in acquainting the reader with higher order terms.

In the third harmonic derivation, the user of this iteration scheme is first exposed to the development of more than one term in the solution for a particular order. In other words, the iteration yields both the leading term to the third harmonic plus the first correction term to the present solution. At the same time, the use of Eq. A.38 is demonstrated for the first time. It should also be pointed out that the methods used in the third harmonic iteration are identical to those used in the higher order iterations, regardless of the number of terms encountered.

Following the step by step procedure then,

$$\psi_3 = \sum_{i=1}^{2} \frac{\partial \xi_i}{\partial a} \frac{\partial \xi_{2-i}}{\partial a} = \frac{\partial \xi_1}{\partial a}\frac{\partial \xi_2}{\partial a} + \frac{\partial \xi_2}{\partial a}\frac{\partial \xi_1}{\partial a} = 2\frac{\partial \xi_1}{\partial a}\frac{\partial \xi_2}{\partial a} \qquad \text{(A.39)}$$

Remember,

$$\xi_1 = \frac{U_{11}}{\omega} \sin K(L-a) \sin \omega t \qquad \text{(A.24)}$$

$$\frac{\partial \xi_1}{\partial a} = -K \frac{U_{11}}{\omega} \cos K(L-a) \sin \omega t \qquad \text{(A.25)}$$

$$\xi_2 = \frac{U_{11}}{4\omega}\left(\frac{M\beta}{2}\right)\frac{1}{H_2} \sin 2K(L-a) \sin(2\omega t + \theta_2) \qquad \text{(A.40)}$$

$$\frac{\partial \xi_2}{\partial a} = -2K \frac{U_{11}}{4\omega}\left(\frac{M\beta}{2}\right)\frac{1}{H_2} \cos 2K(L-a) \sin(2\omega t + \theta_2) \qquad \text{(A.41)}$$

and

$$\bar{U}_{22} = \frac{1}{2}\bar{U}_{11}\left(\frac{M\beta}{2}\right)\frac{1}{H_2} e^{i\theta_2} \qquad \text{(A.32)}$$

so that

$$\psi_3 = 2\left(-K \frac{U_{11}}{\omega} \cos K(L-a) \sin \omega t\right)\left(-\frac{2K}{2\omega}\bar{U}_{22} \cos 2K(L-a) \sin(2\omega t + \theta_2)\right) \qquad \text{(A.42)}$$

or using Eq. A.38,

$$\psi_3 = 2\left(\frac{\bar{U}_{11}}{c_0}\frac{\bar{U}_{22}}{c_0}\right)\left(-\frac{1}{4}\right)\left[\cos 3K(L-a) \cos(3\omega t + \theta_2) - \cos K(L-a) \cos(\omega t + \theta_2)\right]. \qquad \text{(A.43)}$$

Using Eq. A.8 implies

$$A_{33} = -\frac{1}{2}\frac{\bar{U}_{11}}{c_0}\frac{\bar{U}_{22}}{c_0} \tag{A.44}$$

$$A_{31} = \frac{1}{2}\frac{\bar{U}_{11}}{c_0}\frac{\bar{U}_{22}}{c_0} \tag{A.45}$$

and from Eq. A.10,

$$\begin{aligned} u_{33} &= \frac{1}{2}\frac{\bar{U}_{11}}{c_0}\frac{\bar{U}_{22}}{c_0}\, 2\,\frac{1}{2H_3}\,\beta c_0 e^{i\theta_3} \\ &= \frac{1}{2}\bar{U}_{11}\left(\frac{M\beta}{2}\right)^2\frac{1}{H_2H_3}\, e^{i(\theta_2+\theta_3)} \end{aligned} \tag{A.46}$$

and

$$\begin{aligned} u_{31} &= -\frac{1}{2}\frac{\bar{U}_{11}}{c_0}\frac{\bar{U}_{22}}{c_0}\, 2\,\frac{1}{2H_1}\,\beta c_0 e^{i\theta_1} \\ &= -\frac{1}{2}\bar{U}_{11}\left(\frac{M\beta}{2}\right)^2\frac{1}{H_1H_2}\, e^{i(\theta_1+\theta_2)} \end{aligned} \tag{A.47}$$

Now $u_3 = u_{33} + u_{31}$, and when the substitutions are made, the final form of the third harmonic iteration becomes

$$\begin{aligned} u_3 = {} & \frac{1}{2}\bar{U}_{11}\left(\frac{M\beta}{2}\right)^2\frac{1}{H_2H_3}\sin 3K(L-a)\cos(3\omega t+\theta_2+\theta_3) \\ & -\frac{1}{2}\bar{U}_{11}\left(\frac{M\beta}{2}\right)^2\frac{1}{H_1H_2}\sin K(L-a)\cos(\omega t+\theta_1+\theta_2). \end{aligned} \tag{A.48}$$

A combination of the first three orders to obtain the total solution thus far results in

$$
\begin{aligned}
u = {} & \bar{U}_{11} \sin K(L-a) \cos \omega t \\
& + \frac{1}{2} \bar{U}_{11} \left(\frac{M\beta}{2}\right) \frac{1}{H_2} \sin 2K(L-a) \cos(2\omega t + \theta_2) \\
& + \frac{1}{2} \bar{U}_{11} \left(\frac{M\beta}{2}\right)^2 \frac{1}{H_2 H_3} \sin 3K(L-a) \cos(3\omega t + \theta_2 + \theta_3) \\
& - \frac{1}{2} \bar{U}_{11} \left(\frac{M\beta}{2}\right)^2 \frac{1}{H_1 H_2} \sin K(L-a) \cos(\omega t + \theta_1 + \theta_2).
\end{aligned}
\tag{A.49}
$$

Notice that the second term of the third order solution is of order one in j, (with reference to Eq. A.3), and thus it is termed the first correction term to the fundamental or first order solution.

Rewriting Eq. A.49 in order to achieve a form consistent with this last observation and the form of Eq. 2.20 results in

$$
\begin{aligned}
2u/\bar{U}_{11} = {} & \sin K(L-a) \Big\{ 2 \cos \omega t \\
& \qquad - \frac{1}{2} \left(\frac{M\beta}{2}\right)^2 \frac{1}{H_1 H_2} \cos(\omega t + \theta_1 + \theta_2) \\
& \qquad + \cdots \Big\} \\
& + \sin 2K(L-a) \Big\{ \frac{1}{2} \left(\frac{M\beta}{2}\right) \frac{1}{H_2} \cos(\omega t + \theta_2) \\
& \qquad + \cdots \Big\} \\
& + \sin 3K(L-a) \Big\{ \frac{1}{2} \left(\frac{M\beta}{2}\right)^2 \frac{1}{H_2 H_3} \cos(3\omega t + \theta_2 + \theta_3) \\
& \qquad + \cdots \Big\} \\
& + \cdots
\end{aligned}
\tag{A.50}
$$

APPENDIX B

Computer Programs

An IBM-360 digital computer was utilized in performing the calculations, and obtaining the waveforms predicted by the theory. Computer programs were written for each of the basic formulations as well as for each of the variations investigated. Each of these programs, together with a short description, is presented in this section. Included with the description is a list of the symbology peculiar to that particular program.

B-1: Program FINAMPI

This program has written to perform the calculations and plot the graphs for the perturbation solution. It was originally written in FORTRAN 60 language and subsequently converted when the IBM-360 became available.

The program was designed so that the Mach number, resonance frequency, length of the tube, position in the tube, β, $\Delta\omega$, and δ_1 are the input parameters. The first half of the program is devoted to the velocity profiles and the second half to the pressure waveforms. The functions defined at the end of the main program are part of the conversion from FORTRAN 60 to FORTRAN IV. A list of the symbology pertinent to this program, together with the corresponding parameter in the original formulation follows.

Table B.1

D(N) = δ_n

S = Mach number

WD = $\Delta\omega$

WR = ω_r

B = β

F = L

A = a

U = Velocity

P = Pressure

PI = π

WT = ωt

Q(K) = $q_k = 1/2 H_k$

THET(N) = θ_n

Vxyz = the expansion of $(X,y)_z$ for the velocity equation

Yxyz = the expansion of $(X,y)_z$ for the pressure equation

```
      DIMENSION Q(10),THET(10),D(10),U(600),T(600),P(600),WT(600)
      REAL*8 LABEL/4H    /
      REAL*8 ITITLE(12)/'FINAMPI ',' P. G. R','UFF  VEL',
     1 'OCITY DI','STRIBUTI','ON THRU ','SIXTH HA','RMONIC  ',
     2 '      M=','        ','     WD=','        '/
      REAL*8 JTITLE(12)/'FINAMPI ',' P. G. R','UFF  PRE',
     1 'SSURE DI','STRIBUTI','ON THRU ','SIXTH HA','RMONIC  ',
     2 '      M=','        ','     WD=','        '/
      M=1                                                               0004
    2 READ(5,1) D(1),S,WD,WR,B,F,A
    1 FORMAT(2F10.5,5F10.2)
      READ(5,150) JTITLE(10),JTITLE(12)
  150 FORMAT(2A8)
      WRITE(6,44)
   44 FORMAT(1H1,5X,4HD(1),7X,1HS,9X,2HWD,8X,2HWR,9X,1HB,9X,1HF,9X,1HA
     1 ,/)
      WRITE(6,99) D(1),S,WD,WR,B,F,A
   99 FORMAT(2F10.5,5F10.2)                                             0010
      DO3 K=1,10                                                        0011
      TEMP=K                                                            0012
      D(K)=D(1)/SQRTF(TEMP)                                             0013
      Q(K)=1./(2.*SQRT(((D(K)-D(1)+(2.*WD/WR))**2)+(D(K)**2)))
      THET(K) =-ATAN((D(K)-D(1)+2.*(WD/WR))/D(K))
    3 CONTINUE                                                          0016
      WRITE(6,100)
  100 FORMAT(1H0,5X,1HI,12X,4HD(I),11X,4HQ(I),10X,7HTHET(I),/)
      DO73 I=1,10                                                       0017
      WRITE(6,27) I,D(I),Q(I),THET(I)
   27 FORMAT(I6,1F19.7,2F15.7)
   73 CONTINUE                                                          0020
      PI=3.1416                                                         0021
      C=PI/F                                                            0022
      OMEGA=WR+WD                                                       0023
      T(0)=0.                                                           0024
      DO 5 J=1,600                                                      0025
      T(J)=T(J-1)+0.0000272                                             0026
    5 CONTINUE                                                          0027
      DO 88 L=1,600                                                     0028
      WT(L) = OMEGA * T(L)
   88 CONTINUE                                                          0030
C     THE FOLLOWING PORTION COMPUTES AND GRAPHS THE NORMALIZED VELOCITY 0031
C     DISTRIBUTION                                                      0032
      WRITE(6,12)
```

```
12 FORMAT(1H1,4X,1HK,13X,4HT(K),1 X,5HWT(K),12X,4HU(K),/)
   K=0
   DO 7 K=1,600
   V121=Q(1)*Q(2)*COSF(OMEGA*T(K)+THET(1)+THET(2))
   V12231=Q(1)*(Q(2)**2)*Q(3)*COSF(OMEGA*T(K)+THET(1)+(2.*THET(2))
  1 +THET(3))
   V12M23 =Q(1)*(Q(2)**2)*Q(3)*COSF(OMEGA*T(K)+THET(1)+THET(3))
   V11221=(Q(1)**2)*(Q(2)**2)*COSF(OMEGA*T(K)+(2.*THET(1))+
  1 (2.*THET(2)))
   V12M21 =(Q(1)**2)*(Q(2)**2)*COSF(OMEGA*T(K)+2.*THET(1))
   V22=Q(2)*COSF(2.*OMEGA*T(K)+THET(2))
   V2232=(Q(2)**2)*Q(3)*COSF(2.*OMEGA*T(K)+(2.*THET(2))+THET(3))
   V1222=Q(1)*(Q(2)**2)*COSF(2.*OMEGA*T(K)+THET(1)+(2.*THET(2)))
   V233=Q(2)*Q(3)*COSF(3.*OMEGA*T(K)+THET(2)+THET(3))
   V23343=Q(2)*(Q(3)**2)*Q(4)*COSF(3.*OMEGA*T(K)+THET(2)+
  1 (2.*THET(3))+THET(4))
   V22343=(Q(2)**2)*Q(3)*Q(4)*COSF(3.*OMEGA*T(K)+(2.*THET(2))+
  1 THET(3)+THET(4))
   V22333=(Q(2)**2)*(Q(3)**2)*COSF(3.*OMEGA*T(K)+(2.*THET(2))+
  1 (2.*THET(3)))
   V12233=Q(1)*(Q(2)**2)*Q(3)*COSF(3.*OMEGA*T(K)+THET(1)+
  1 (2.*THET(2))+THET(3))
   V2344=Q(2)*Q(3)*Q(4)*COSF(4.*OMEGA*T(K)+THET(2)+THET(3)+THET(4))
   V2244=(Q(2)**2)*Q(4)*COSF(4.*OMEGA*T(K)+(2.*THET(2))+THET(4))
   V23455=Q(2)*Q(3)*Q(4)*Q(5)*COSF(5.*OMEGA*T(K)+THET(2)+THET(3)+
  1 THET(4)+THET(5))
   V22455=(Q(2)**2)*Q(4)*Q(5)*COSF(5.*OMEGA*T(K)+(2.*THET(2))+
  1 THET(4)+THET(5))
   V22355=(Q(2)**2)*Q(3)*Q(5)*COSF(5.*OMEGA*T(K)+(2.*THET(2))+
  1 THET(3)+THET(5))
   V22334=(Q(2)**2)*(Q(3)**2)*Q(4)*COSF(2.*OMEGA*T(K)+(2.*THET(2))+
  1 (2.*THET(3))+THET(4))
   V22234=(Q(2)**3)*Q(3)*Q(4)*COSF(2.*OMEGA*T(K)+(3.*THET(2))+
  1 THET(3)+THET(4))
   V22233=(Q(2)**3)*(Q(3)**2)*COSF(2.*OMEGA*T(K)+(3.*THET(2))+
  1 (2.*THET(3)))
   V12223=Q(1)*(Q(2)**3)*Q(3)*COSF(2.*OMEGA*T(K)+THET(1)+(3.*THET(2))
  1 +THET(3))
   V11222=(Q(1)**2)*(Q(2)**3)*COSF(2.*OMEGA*T(K)+(2.*THET(1))+
  1 (3.*THET(2)))
   V22M23 =Q(1)*(Q(2)**3)*Q(3)*COSF(2.*OMEGA*T(K) +THET(1)+THET(2)+
  1 THET(3))
   V112M2 =(Q(1)**2)*(Q(2)**3)*COSF(2.*OMEGA*T(K) +(2.*THET(1))+
1 THET(2))
 V2M223 =(Q(2)**3)*Q(3)*Q(4)*COSF(2.*OMEGA*T(K)+THET(2)+THET(3)+
1 THET(4))
 V222M2 =(Q(2)**4)*Q(4)*COSF(2.*OMEGA*T(K)+(2.*THET(2))+THET(4))
 V12M22 =Q(1)*(Q(2)**3)*Q(3)*COSF(2.*OMEGA*T(K)-THET(1)+THET(2)+
1 THET(3))
 V11222=(Q(1)**2)*(Q(2)**3)*COSF(2.*OMEGA*T(K)+(2.*THET(1))+
1 (3.*THET(2)))
 V23445=Q(2)*Q(3)*(Q(4)**2)*Q(5)*COSF(4.*OMEGA*T(K)+THET(2)+
1 THET(3)+(2.*THET(4))+THET(5))
 V22445=(Q(2)**2)*(Q(4)**2)*Q(5)*COSF(4.*OMEGA*T(K)+(2.*THET(2))+
1 (2.*THET(4))+THET(5))
 V22345=(Q(2)**2)*Q(3)*Q(4)*Q(5)*COSF(4.*OMEGA*T(K)+(2.*THET(2))+
1 THET(3)+THET(4)+THET(5))
 V23344=Q(2)*(Q(3)**2)*(Q(4)**2)*COSF(4.*OMEGA*T(K)+THET(2)+
1 (2.*THET(3))+(2.*THET(4)))
 V22344=(Q(2)**2)*Q(3)*(Q(4)**2)*COSF(4.*OMEGA*T(K)+(2.*THET(2))+
1 THET(3)+(2.*THET(4)))
 V22334=(Q(2)**2)*(Q(3)**2)*Q(4)*COSF(4.*OMEGA*T(K)+(2.*THET(2))+
1 (2.*THET(3))+THET(4))
 V12234=Q(1)*(Q(2)**2)*Q(3)*Q(4)*COSF(4.*OMEGA*T(K)+THET(1)+
1 (2.*THET(2))+THET(3)+THET(4))
 V22234=(Q(2)**3)*Q(3)*Q(4)*COSF(4.*OMEGA*T(K)+(3.*THET(2))+THET(3)
1 +THET(4))
 V12224=Q(1)*(Q(2)**3)*Q(4)*COSF(4.*OMEGA*T(K)+THET(1)+(3.*THET(2))
1 +THET(4))
 V23456=Q(2)*Q(3)*Q(4)*Q(5)*Q(6)*COSF(6.*OMEGA*T(K)+THET(2)+
1 THET(3)+THET(4)+THET(5)+THET(6))
 V22456=(Q(2)**2)*Q(4)*Q(5)*Q(6)*COSF(6.*OMEGA*T(K)+(2.*THET(2))+
1 THET(4)+THET(5)+THET(6))
 V22356=(Q(2)**2)*Q(3)*Q(5)*Q(6)*COSF(6.*OMEGA*T(K)+(2.*THET(2))+
1 THET(3)+THET(5)+THET(6))
 V22346=(Q(2)**2)*Q(3)*Q(4)*Q(6)*COSF(6.*OMEGA*T(K)+(2.*THET(2))
1 +THET(3)+THET(4)+THET(6))
 V22246=(Q(2)**3)*Q(4)*Q(6)*COSF(6.*OMEGA*T(K)+(3.*THET(2))+
1 THET(4)+THET(6))
 V22336=(Q(2)**2)*(Q(3)**2)*Q(6)*COSF(6.*OMEGA*T(K)+(2.*THET(2))+
1 (2.*THET(3))+THET(6))
 U(K)=0.5*((SINF(C*(F-A)))*(2.*COSF(OMEGA*T(K))-((S*B)**2)*V121+
1 ((S*B)**4)*(V12231-(0.5)*V12M23 +V11221+(0.5)*(V12M21)))+
2 (SINF(2.*C*(F-A)))*((S*B)*V22-((S*B)**3)*(V2232+V1222)+
3 ((S*B)**5)*(V22334+(0.25)*V22234+V22233+(2.5)*V12223+V11222-
4 (0.5)*V22M23 +(0.5)*V112M2 -(0.5)*V2M223 -(0.125)*V222M2 +
     5 (0.5)*V12M22 +(0.25)*V11222))+(SINF(3.*C*(F-A)))*(((S*B)**2)*        0118
     6 V233-((S*B)**4)*(V23343+(0.25)*V22343+V22333+(1.5)*V12233))+         0119
     7 (SINF(4.*C*(F-A)))*(((S*B)**3)*(V2344+(0.25)*V2244)-((S*B)**5)       0120
     8 *(V23445+(0.25)*V22445+(0.5)*V22345+V23344+(0.25)*V22344+            0121
     9 V22334+(2.)*V12234+(0.5)*V22234+V12224)))                            0122
      W=0.5*((SINF(5.*C*(F-A)))*(((S*B)**4)*(V23455+(0.25)*V22455+          0123
     1 (0.5)*V22355))+(SINF(6.*C*(F-A)))*(((S*B)**5)*(V23456+(0.25)*        0124
     2 V22456+(0.5)*V22356+(0.5)*V22346+(0.125)*V22246+(0.25)*V22336)))     0125
      U(K)=U(K)+W                                                           0126
      WRITE(6,75) K,T(K),WT(K),U(K)
   75 FORMAT(I6,1F19.7,2F15.7)
    7 CONTINUE                                                              0129
      CALL DRAW(600,WT,U,0,0,LABEL,ITITLE,0,0,0,0,1,1,9,15,0,1)
C     THE FOLLOWING PORTION COMPUTES AND GRAPHS THE NORMALIZED              0138
C     PRESSURE DISTRIBUTION                                                 0139
      WRITE(6,77)
   77 FORMAT(1H1,4X,1HL,13X,4HT(L),10X,5HWT(L),12X,4HP(L),/)
      L=0                                                                   0140
      DO 9 L=1,600                                                          0141
      Y121=Q(1)*Q(2)*SINF(OMEGA*T(L)+THET(1)+THET(2))                       0142
      Y12231=Q(1)*(Q(2)**2)*Q(3)*SINF(OMEGA*T(L)+THET(1)+(2.*THET(2))       0143
     1 +THET(3))                                                            0144
      Y12M23 =Q(1)*(Q(2)**2)*Q(3)*SINF(OMEGA*T(L)+THET(1)+THET(3))          0145
      Y11221=(Q(1)**2)*(Q(2)**2)*SINF(OMEGA*T(L)+(2.*THET(1))+              0146
     1 (2.*THET(2)))                                                        0147
      Y12M21 =(Q(1)**2)*(Q(2)**2)*SINF(OMEGA*T(L)+2.*THET(1))               0148
      Y22=Q(2)*SINF(2.*OMEGA*T(L)+THET(2))                                  0149
      Y2232=(Q(2)**2)*Q(3)*SINF(2.*OMEGA*T(L)+(2.*THET(2))+THET(3))         0150
      Y1222=Q(1)*(Q(2)**2)*SINF(2.*OMEGA*T(L)+THET(1)+(2.*THET(2)))         0151
      Y233=Q(2)*Q(3)*SINF(3.*OMEGA*T(L)+THET(2)+THET(3))                    0152
      Y23343=Q(2)*(Q(3)**2)*Q(4)*SINF(3.*OMEGA*T(L)+THET(2)+                0153
     1 (2.*THET(3))+THET(4))                                                0154
      Y22343=(Q(2)**2)*Q(3)*Q(4)*SINF(3.*OMEGA*T(L)+(2.*THET(2))+           0155
     1 THET(3)+THET(4))                                                     0156
      Y22333=(Q(2)**2)*(Q(3)**2)*SINF(3.*OMEGA*T(L)+(2.*THET(2))+           0157
     1 (2.*THET(3)))                                                        0158
      Y12233=Q(1)*(Q(2)**2)*Q(3)*SINF(3.*OMEGA*T(L)+THET(1)+                0159
     1 (2.*THET(2))+THET(3))                                                0160
      Y2344=Q(2)*Q(3)*Q(4)*SINF(4.*OMEGA*T(L)+THET(2)+THET(3)+THET(4))      0161
      Y2244=(Q(2)**2)*Q(4)*SINF(4.*OMEGA*T(L)+(2.*THET(2))+THET(4))         0162
      Y23455=Q(2)*Q(3)*Q(4)*Q(5)*SINF(5.*OMEGA*T(L)+THET(2)+THET(3)+        0163
     1 THET(4)+THET(5))                                                     0164
      Y22455=(Q(2)**2)*Q(4)*Q(5)*SINF(5.*OMEGA*T(L)+(2.*THET(2))+           0165
1 THET(4)+THET(5))
 Y22355=(Q(2)**2)*Q(3)*Q(5)*SINF(5.*OMEGA*T(L)+(2.*THET(2))+
1 THET(3)+THET(5))
 Y22334=(Q(2)**2)*(Q(3)**2)*Q(4)*SINF(2.*OMEGA*T(L)+(2.*THET(2))+
1 (2.*THET(3))+THET(4))
 Y22234=(Q(2)**3)*Q(3)*Q(4)*SINF(2.*OMEGA*T(L)+(3.*THET(2))+
1 THET(3)+THET(4))
 Y22233=(Q(2)**3)*(Q(3)**2)*SINF(2.*OMEGA*T(L)+(3.*THET(2))+
1 (2.*THET(3)))
 Y12223=Q(1)*(Q(2)**3)*Q(3)*SINF(2.*OMEGA*T(L)+THET(1)+(3.*THET(2))
1 +THET(3))
 Y11222=(Q(1)**2)*(Q(2)**3)*SINF(2.*OMEGA*T(L)+(2.*THET(1))+
1 (3.*THET(2)))
 Y22M23 =Q(1)*(Q(2)**3)*Q(3)*SINF(2.*OMEGA*T(L) +THET(1)+THET(2)+
1 THET(3))
 Y112M2 =(Q(1)**2)*(Q(2)**3)*SINF(2.*OMEGA*T(L) +(2.*THET(1))+
1 THET(2))
 Y2M223 =(Q(2)**3)*Q(3)*Q(4)*SINF(2.*OMEGA*T(L) +THET(2)+THET(3)+
1 THET(4))
 Y222M2 =(Q(2)**4)*Q(4)*SINF(2.*OMEGA*T(L)+(2.*THET(2))+THET(4))
 Y12M22 =Q(1)*(Q(2)**3)*Q(3)*SINF(2.*OMEGA*T(L)-THET(1)+THET(2)+
1 THET(3))
 Y11222=(Q(1)**2)*(Q(2)**3)*SINF(2.*OMEGA*T(L)+(2.*THET(1))+
1 (3.*THET(2)))
 Y23445=Q(2)*Q(3)*(Q(4)**2)*Q(5)*SINF(4.*OMEGA*T(L)+THET(2)+
1 THET(3)+(2.*THET(4))+THET(5))
 Y22445=(Q(2)**2)*(Q(4)**2)*Q(5)*SINF(4.*OMEGA*T(L)+(2.*THET(2))+
1 (2.*THET(4))+THET(5))
 Y22345=(Q(2)**2)*Q(3)*Q(4)*Q(5)*SINF(4.*OMEGA*T(L)+(2.*THET(2))+
1 THET(3)+THET(4)+THET(5))
 Y23344=Q(2)*(Q(3)**2)*(Q(4)**2)*SINF(4.*OMEGA*T(L)+THET(2)+
1 (2.*THET(3))+(2.*THET(4)))
 Y22344=(Q(2)**2)*Q(3)*(Q(4)**2)*SINF(4.*OMEGA*T(L)+(2.*THET(2))+
1 THET(3)+(2.*THET(4)))
 Y22334=(Q(2)**2)*(Q(3)**2)*Q(4)*SINF(4.*OMEGA*T(L)+(2.*THET(2))+
1 (2.*THET(3))+THET(4))
 Y12234=Q(1)*(Q(2)**2)*Q(3)*Q(4)*SINF(4.*OMEGA*T(L)+THET(1)+
1 (2.*THET(2))+THET(3)+THET(4))
 Y22234=(Q(2)**3)*Q(3)*Q(4)*SINF(4.*OMEGA*T(L)+(3.*THET(2))+THET(3)
1 +THET(4))
 Y12224=Q(1)*(Q(2)**3)*Q(4)*SINF(4.*OMEGA*T(L)+THET(1)+(3.*THET(2))
1 +THET(4))
 Y23456=Q(2)*Q(3)*Q(4)*Q(5)*Q(6)*SINF(6.*OMEGA*T(L)+THET(2)+
     1 THET(3)+THET(4)+THET(5)+THET(6))                                    0209
      Y22456=(Q(2)**2)*Q(4)*Q(5)*Q(6)*SINF(6.*OMEGA*T(L)+(2.*THET(2))+      0210
     1 THET(4)+THET(5)+THET(6))                                            0211
      Y22356=(Q(2)**2)*Q(3)*Q(5)*Q(6)*SINF(6.*OMEGA*T(L)+(2.*THET(2))+      0212
     1 THET(3)+THET(5)+THET(6))                                            0213
      Y22346=(Q(2)**2)*Q(3)*Q(4)*Q(6)*SINF(6.*OMEGA*T(L)+(2.*THET(2))+      0214
     1 THET(3)+THET(4)+THET(6))                                            0215
      Y22246=(Q(2)**3)*Q(4)*Q(6)*SINF(6.*OMEGA*T(L)+(3.*THET(2))+           0216
     1 THET(4)+THET(6))                                                    0217
      Y22336=(Q(2)**2)*(Q(3)**2)*Q(6)*SINF(6.*OMEGA*T(L)+(2.*THET(2))+      0218
     1 (2.*THET(3))+THET(6))                                               0219
      P(L)=0.5*(2.*SINF(OMEGA*T(L))-((S*B)**2)*Y121+((S*B)**4)*(Y12231      0220
     1 -(0.5)*Y12M23 +Y11221+(0.5)*Y12M21 )+(S*B)*Y22-((S*B)**3)*           0221
     2 (Y2232+Y1222)+((S*B)**5)*(Y22334+(0.25)*Y22234+Y22233+(2.5)*         0222
     3 Y12223+Y11222-(0.5)*Y22M23 +(0.5)*Y112M2 -(0.5)*Y2M223 -(0.125)      0223
     4 *Y222M2 +(0.5)*Y12M22 +(0.25)*Y11222)+((S*B)**2)*Y233-              0224
     5 ((S*B)**4)*(Y23343+(0.25)*Y22343+Y22333+(1.5)*Y12233)+              0225
     6 ((S*B)**3)*(Y2344+(0.25)*Y2244)-((S*B)**5)*(Y23445+(0.25)*           0226
     7 Y22445+(0.5)*Y22345+Y23344+(0.25)*Y22344+Y22334+(2.)*Y12234+         0227
     8 (0.5)*Y22234+Y12224)+((S*B)**4)*(Y23455+(0.25)*Y22455+(0.5)*         0228
     9 Y22355))                                                            0229
      R=((S*B)**5)*(Y23456+(0.25)*Y22456+(0.5)*Y22356+(0.5)*Y22346          0230
     1 +(0.125)*Y22246+(0.25)*Y22336)*(0.5)                                0231
      P(L)=P(L)+R                                                          0232
      WRITE(6,33) L,T(L),WT(L),P(L)
   33 FORMAT(I6,1F19.7,2F15.7)
    9 CONTINUE                                                             0235
      CALL DRAW(600,WT,P,0,0,LABEL,JTITLE,1.20E+00,4.00E-01,
     1 0,0,0,0,9,15,0,L)
      CONTINUE                                                             0244
      IF(M-3)30,40,40                                                      0245
   30 M=M+1                                                                0246
      GO TO 2                                                              0247
   40 STOP
      END
```

```
FUNCTION COSF(X)
COSF = COS(X)
END



FUNCTION SINF(X)
SINF = SIN(X)
END



FUNCTION SQRTF(X)
SQRTF = SQRT(X)
END
```

B-2: Program FINAMPIZ

This program was designed to duplicate the output of FINAMPI using the Fourier synthesis formulation through sixth order terms. Since the final results of the perturbation approach are in the same form as those of the Fourier synthesis approach the program FINAMPI was converted to FINAMPIZ by setting the corrective terms to zero. Because FINAMPIZ is essentially FINAMPI, Table B.1 is applicable here.

```
      DIMENSICN C(10),THET(10),D(10),U(600),T(600),P(600),WT(600)
      REAL*8 LABEL74F    /
      REAL*8 JTITLE(12)/'FINAMPIX',' P. G. R','UFF  PRE',
     1 'SSURE DI','STRIBUTI','ON THRU ','SIXTH HA','RMONIC  ',
     2 '    M=','        ','    WD=','        '/
      M=1                                                          0004
    2 READ(5,1) C(1),S,WD,WR,B,F,A
    1 FCRMAT(2F1C.5,5F10.2)
      READ(5,150) JTITLE(1C),JTITLE(12)
  150 FCRMAT(2A8)
      WRITE(6,44)
   44 FCRMAT(1H1,5X,4HC(1),7X,1HS,9X,2HWD,8X,2HWR,9X,1HB,9X,1HF,9X,1HA
     1 ,/)
      WRITE(6,99) C(1),S,WD,WR,B,F,A
   99 FCRMAT(2F10.5,5F10.2)                                        0010
      CC3 K=1,10                                                   0011
      TEMP=K                                                       0012
      C(K)=C(1)/SCRTF(TEMP)                                        0013
      C(K)=1./(2.*SCRT(((C(K)-D(1)+(2.*WD/WR))**2)+(D(K)**2)))
      THET(K) =-ATAN((D(K)-D(1)+2.*(WD/WR))/D(K))
    3 CCNTINUE                                                     0016
      WRITE(6,10C)
  100 FCRMAT(1HC,5X,1HI,12X,4HD(I),11X,4HQ(I),10X,7HTHET(I),/)
      DC73 I=1,1C                                                  0017
      WRITE(6,27) I,D(I),C(I),THET(I)
   27 FCRMAT(I6,1F19.7,2F15.7)
   73 CCNTINUE                                                     0020
      PI=3.1416                                                    0021
      C=PI/F                                                       0022
      OMEGA=WR+WC                                                  0023
      T(C)=C.                                                      0024
      CC 5 J=1,6CC                                                 0025
      T(J)=T(J-1)+C.CCCC272                                        0026
    5 CCNTINUE                                                     0027
      CC 88 L=1,6CC                                                0028
      WT(L) = CMEGA * T(L)
   88 CCNTINUE                                                     0030
      WRITE(6,77)
   77 FCRMAT(1H1,4X,1HL,13X,4HT(L),10X,5HWT(L),12X,4HP(L),/)
      L=0                                                          0140
      CC 9 L=1,600                                                 0141
      Y121=C.0
      Y12231=C.C
      Y12M23=0.0
      Y11221=0.0
      Y12M21=0.0
      Y22=Q(2)*SINF(2.*OMEGA*T(L)+THET(2))                                  0149
      Y2232=0.0
      Y1222=0.0
      Y233=Q(2)*Q(3)*SINF(3.*OMEGA*T(L)+THET(2)+THET(3))                    0152
      Y23343=0.0
      Y22343=0.0
      Y22333=0.0
      Y12233=0.0
      Y2344=Q(2)*Q(3)*Q(4)*SINF(4.*OMEGA*T(L)+THET(2)+THET(3)+THET(4))      0161
      Y2244=(Q(2)**2)*Q(4)*SINF(4.*OMEGA*T(L)+(2.*THET(2))+THET(4))         0162
      Y23455=Q(2)*Q(3)*Q(4)*Q(5)*SINF(5.*OMEGA*T(L)+THET(2)+THET(3)+        0163
     1 THET(4)+THET(5))                                                     0164
      Y22455=(Q(2)**2)*Q(4)*Q(5)*SINF(5.*OMEGA*T(L)+(2.*THET(2))+           0165
     1 THET(4)+THET(5))                                                     0166
      Y22355=(Q(2)**2)*Q(3)*Q(5)*SINF(5.*OMEGA*T(L)+(2.*THET(2))+           0167
     1 THET(3)+THET(5))                                                     0168
      Y22334=0.0
      Y22234=0.0
      Y22233=0.0
      Y12223=0.0
      Y11222=0.0
      Y22M23=0.0
      Y112M2=0.0
      Y2M223=0.0
      Y222M2=0.0
      Y12M22=0.0
      Y11222=0.0
      Y23445=0.0
      Y22445=0.0
      Y22345=0.0
      Y23344=0.0
      Y22344=0.0
      Y22334=0.0
      Y12234=0.0
      Y22234=0.0
      Y12224=0.0
      Y23456=Q(2)*Q(3)*Q(4)*Q(5)*Q(6)*SINF(6.*OMEGA*T(L)+THET(2)+           0208
     1 THET(3)+THET(4)+THET(5)+THET(6))                                     0209
      Y22456=(Q(2)**2)*Q(4)*Q(5)*Q(6)*SINF(6.*OMEGA*T(L)+(2.*THET(2))+      0210
     1 THET(4)+THET(5)+THET(6))                                             0211
```

```
      Y22356=(Q(2)**2)*Q(3)*Q(5)*Q(6)*SINF(6.*OMEGA*T(L)+(2.*THET(2))+        0212
     1 THET(3)+THET(5)+THET(6))                                                0213
      Y22346=(Q(2)**2)*Q(3)*Q(4)*Q(6)*SINF(6.*OMEGA*T(L)+(2.*THET(2))+        0214
     1 THET(3)+THET(4)+THET(6))                                                0215
      Y22246=(Q(2)**3)*Q(4)*Q(6)*SINF(6.*CMEGA*T(L)+(3.*THET(2))+             0216
     1 THET(4)+THET(6))                                                        0217
      Y22336=(C(2)**2)*(Q(3)**2)*Q(6)*SINF(6.*OMEGA*T(L)+(2.*THET(2))+        0218
     1 (2.*THET(3))+THET(6))                                                   0219
      P(L)=0.5*(2.*SINF(CMEGA*T(L))-((S*B)**2)*Y121+((S*B)**4)*(Y12231        0220
     1 -(0.5)*Y12M23 +Y11221+(0.5)*Y12M21 )+(S*B)*Y22-((S*B)**3)*             0221
     2 (Y2232+Y1222)+((S*B)**5)*(Y22334+(0.25)*Y22234+Y22233+(2.5)*,          0222
     3 Y12223+Y11222-(C.5)*Y22M23 +(0.5)*Y112M2 -(0.5)*Y2M223 -(0.125)       0223
     4 *Y222M2 +(C.5)*Y12M22 +(0.25)*Y11222)+((S*B)**2)*Y233-                  0224
     5 ((S*B)**4)*(Y23343+(0.25)*Y22343+Y22333+(1.5)*Y12233)+                  0225
     6 ((S*B)**3)*(Y2344+(C.25)*Y2244)-((S*B)**5)*(Y23445+(0.25)*             0226
     7 Y22445+(0.5)*Y22345+Y23344+(0.25)*Y22344+Y22334+(2.)*Y12234+           0227
     8 (C.5)*Y22234+Y12224)+((S*B)**4)*(Y23455+(0.25)*Y22455+(0.5)*           0228
     9 Y22355))                                                                0229
      R=((S*B)**5)*(Y23456+(C.25)*Y22456+(0.5)*Y22356+(0.5)*Y22346            0230
     1 +(0.125)*Y22246+(0.25)*Y22336)*(0.5)                                    0231
      P(L)=P(L)+R                                                              0232
      WRITE(6,33) L,T(L),WT(L),P(L)
   33 FCRMAT(I6,1F19.7,2F15.7)
    9 CCNTINUE                                                                 0235
      CALL CRAW(60C,WT,P,0,0,LABEL,JTITLE,1.20E+00,4.00E-01,
     1 C,0,0,0,9,15,C,L)
      CCNTINUE                                                                 0244
      IF(M-3)30,40,4C                                                          0245
   30 M=M+1                                                                    0246
      GC TO 2                                                                  0247
   40 STCP
      END
```

```
FUNCTION COSF(X)
COSF = COS(X)
END
```

```
FUNCTION SINF(X)
SINF = SIN(X)
END
```

```
FUNCTION SQRTF(X)
SQRTF = SQRT(X)
END
```

B-3: Program QCURVES

This program was designed to compute and graph the normalized Q-curve for each of the six harmonics using the perturbation solution.

In this program, the input parameters consist of the resonant frequency, the tube length, δ_1, β, and the piston acceleration. The symbology pertinent to this program is given in the following table.

Table B.2

AO = piston acceleration

F = length of tube

Delt(N) = δ_n

H(N) = hypothenuse of triangle described in Fig. A.1

THET(N) = θ_n

SCRIPN(M) = unnormalized magnitude of the n-th harmonic at point m

MAXA = the normalizing factor defined for each harmonic

SMALA = defined subroutine to main program

PHI = defined subprogram to main program

```
C
C
C     THIS PROGRAM IS DESIGNED TO COMPUTE AND GRAPH THE
C     Q CURVE FOR EACH HARMONIC.  EACH CURVE IS NORMALIZED
C     TO APPROXIMATELY ONE. THIS IS ACCOMPLISHED BY DIVIDING
C     THE COMPUTED MAGNITUDE OF EACH HARMONIC BY THE MAGNITUDE
C     OF THE FIRST TERM OF THAT HARMONIC.
C
C
      DIMENSION DELT(6),THET(7),H(7),SCRIPA(81),SCRIPB(81),SCRIPC(81),
     1 SCRIPD(81),SCRIPE(81),SCRIPF(81),MAXA(81),WD(81),QA(81),QB(81),
     2 QC(81),QD(81),QE(81),QF(81),FACT(6,81),PHASE(6,81)
      COMMON THET,H,AMPF
      REAL MACH
      REAL MAXA
      REAL*8 ITITLE(12)/'QCURVES ','P. G. RU','FF CURVE',' FOR THE'
     1 ,'        ','HARMONIC','        ','   AO = ','        ',
     2 '        ','        ','        '/
      READ(5,10) DELT(1),WR,BETA,X,AO,F
   10 FORMAT(6F10.5)
      WRITE(6,15)
   15 FORMAT(1H1,2X,7HDELT(1),5X,2HWR,7X,4HBETA,7X,1HX,8X,2HAO,9X,
     1 1HF,/)
      WRITE(6,25) DELT(1),WR,BETA,X,AO,F
   25 FORMAT(6F10.5)
      PI=3.1416
      CO = 343.4
      C=PI/F
      AMPF=(AO*BETA)/(X*PI*WR*CO)
      DO 17 I=1,6
      TEMP=I
      DELT(I)=DELT(1)/SQRT(TEMP)
   17 CONTINUE
      WRITE(6,175)
  175 FORMAT(1H0,10X,1HI,10X,1HC,10X,4HAMPF,10X,7HDELT(I),/)
      DO 57 I=1,6
      WRITE(6,185) I,C,AMPF,DELT(I)
  185 FORMAT(I11,F16.5,F13.10,F15.7)
   57 CONTINUE
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE FIRST HARMONIC
```

```
      DO 9 M=1,81
      WD(M)=-5.0+(M-1)*0.125
      DO 3 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(M)*WR)-(WD(M)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(M)/WR))**2)+(DELT(K)**2))
      MACH=(2.*AO)/(X*PI*WR*CO*H(1))
    3 CONTINUE
      FACT(1,M)=(((MACH*BETA)**1)*SQRT(1.))/(DELT(1)**1)
      WRITE(6,230)
  230 FORMAT(1H0,4X,1HK,10X,1HA,18X,1HB,16X,7HTHET(K),15X,4HH(K),
     1 18X,4HMACH,/)
      DO 211 K=1,6
      WRITE(6,220) K,A,B,THET(K),H(K),MACH
  220 FORMAT(I5,5F20.7)
  211 CONTINUE
      FI = SMALA(-0.5,2,1,3,2,1,7,1,7,1,7,7) * SIN(
     1 PHI(1,1,1,2,1,7,1,7,7,7))+SMALA(0.5,4,1,5,2,2,3,1,7,1,7,7)*SIN
     2  (PHI(1,1,2,2,1,3,1,7,7,7))+ SMALA(-0.25,4,1,5,2,2,3,1,7,
     3 1,7,7) * SIN(PHI(1,1,1,3,1,7,1,7,7,7))
     4 + SMALA(0.5,4,1,6,2,2,7,1,7,1,7,7)*SIN(
     5 PHI(2,1,1,2,1,7,1,7,7,7))+ SMALA(0.25,4,1,6,2,2,7,1,7,1,7,7)*SIN
     6  (PHI(2,1,1,7,1,7,1,7,7,7))
      FII=1+SMALA(-0.5,2,1,3,2,1,7,1,7,1,7,7)*
     1 COS(PHI(1,1,1,2,1,7,1,7,7,7))+ SMALA(0.5,4,1,5,2,2,
     2 3,1,7,1,7,7)*COS(PHI(1,1,2,2,1,3,1,7,7,7)) +
     3 SMALA(-0.25,4,1,5,2,2,3,1,7,1,7,7)* COS(PHI(
     4 1,1,1,3,1,7,1,7,7,7))+ SMALA(0.5,4,1,6,2,2,7,1,7,1,7,7)
     5 * COS(PHI(2,1,1,2,1,7,1,7,7,7)) + SMALA(0.25,4,
     6 1,6,2,2,7,1,7,1,7,7) * COS(PHI(2,1,1,7,1,7,1,7,7,7))
      WRITE(6,210)
  210 FORMAT(1H0,5X,1HM,10X,2HFI,18X,3HFII,17X,9HFACT(1,M),/)
      WRITE(6,200) M,FI,FII,FACT(1,M)
  200 FORMAT(I6,3F20.5)
      SCRIPA(M)=SQRT(FI**2+FII**2)
      SCRIPA(M)=COS(THET(1))*SCRIPA(M)
      MAXA(M)=1.
      QA(M)=SCRIPA(M)/MAXA(M)
    9 CONTINUE
      WRITE(6,55)
```

```
55 FORMAT(1H1,4X,1HM,8X,5HWD(M),5X,9HSCRIPA(M),7X,7HMAXA(M),9X,
  1 5HQA(M),/)
    DC 11 M=1,81
    WRITE(6,65) M,WD(M),SCRIPA(M),MAXA(M),QA(M)
65 FCRMAT(I6,4F20.8)
11 CCNTINUE
    READ(5,91) ITITLE(5),ITITLE(9)
91 FORMAT(2A8)
    REAL*8 ACURVE/8H     FRST/
C
C
C   THIS PORTICN COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C   FOR THE SECCNC HARMONIC.
C
C
    DO 13 J=1,81
    WD(J)=-5.0+(J-1)*0.125
    DO 59 K=1,6
    A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(J)*WR)-(WD(J)**2)
    B=DELT(K) * (WR**2)
    THET(K)=ATAN2(A,B)
    H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(J)/WR))**2)
  1 +(DELT(K)**2))
    MACH = (2.*AC)/(X*PI*WR*CO*H(1))
59 CONTINUE
    FACT(2,J)=(((MACH*BETA)**2)*SQRT(2.))/(DELT(1)**2)
    GA=SMALA(0.5,1,1,1,2,1,7,1,7,1,7,7) * SIN(PHI
  1 (1,2,1,7,1,7,1,7,7,7))-SMALA(0.5,3,1,3,2,2,3,1,7,1,7,7)
  2 * SIN(PHI(2,2,1,3,1,7,1,7,7,7)) -SMALA(0.5,3,1,4,
  3 2,2,7,1,7,1,7,7)* SIN(PHI(1,1,2,2,1,7,1,7,7,7)) +
  4 SMALA(0.5,5,1,5,2,2,3,2,4,1,7,7) * SIN(PHI(2,2,2,3,1,4,
  5 1,7,7,7)) + SMALA(0.125,5,1,5,2,3,3,1,4,1,7,7)*
  6 SIN(PHI(3,2,1,3,1,4,1,7,7,7))
    GB = SMALA(0.5,5,1,5,2,3,3,2,7,1,7,7) * SIN(PHI
  1 (3,2,2,3,1,7,1,7,7,7))+ SMALA(0.5,5,1,6,2,3,3,1,
  2 7,1,7,7) * SIN(PHI(1,1,3,2,1,3,1,7,7,7)) +
  3 SMALA(0.25,5,1,6,2,3,3,1,7,1,7,7) * SIN(PHI(1,1,
  4 3,2,3,1,7,1,7,7)) + SMALA(0.50,5,1,6,2,3,3,1,7,1,
  5 7,7) * SIN(PHI(3,2,1,3,1,7,1,7,7,7)) + SMALA(0.5,5,
  6 1,7,2,3,7,1,7,1,7,7) * SIN(PHI(2,1,3,2,1,7,1,7,7,7))
    GC=-SMALA(0.25,5,1,6,2,3,3,1,7,1,7,7) * SIN(PHI(
  1 1,1,1,2,1,3,1,7,7,7)) + SMALA(0.25,5,1,7,2,3,7,1,7,1,
  2 7,7) * SIN(PHI(2,1,1,2,1,7,1,7,7,7)) - SMALA(0.25,5,
     3 1,5,2,3,3,1,4,1,7,7) * SIN(PHI(1,2,1,3,1,4,1,7,7,7))
     4 - SMALA(0.0625,5,1,5,2,4,4,1,7,1,7,7) * SIN (PHI(
     5 2,2,1,4,1,7,1,7,7,7)) + SMALA(0.25,5,1,6,2,3,3,1,7,1,7,7)
     6 * SIN(PHI(-1,1,1,2,1,3,1,7,7,7)) + SMALA(0.125,5,1,7,
     7 2,3,7,1,7,1,7,7) * SIN(PHI(2,1,3,2,1,7,1,7,7,7))
      GI=GA+GB+GC
      GD=SMALA(0.5,1,1,1,2,1,7,1,7,1,7,7) * COS(PHI(
     1 1,2,1,7,1,7,1,7,7,7)) - SMALA(0.5,3,1,3,2,2,3,1,7,1,7,7)
     6 * COS(PHI(2,2,1,3,1,7,1,7,7,7)) - SMALA(0.5,3,1,4,
     3 2,2,7,1,7,1,7,7)*COS(PHI(1,1,2,2,1,7,1,7,7,7)) +
     4 SMALA(0.5,5,1,5,2,2,3,2,4,1,7,7) * COS(PHI(2,2,2,3,1,4,
     5 1,7,7,7)) + SMALA(0.125,5,1,5,2,3,3,1,4,1,7,7) *
     6 COS(PHI(3,2,1,3,1,4,1,7,7,7))
      GE=SMALA(0.5,5,1,5,2,3,3,2,7,1,7,7) * COS(PHI(
     1 3,2,2,3,1,7,1,7,7,7)) + SMALA(0.5,5,1,6,2,3,3,1,
     2 7,1,7,7) * COS(PHI(1,1,3,2,1,3,1,7,7,7)) + SMALA(0.25,
     3 5,1,6,2,3,3,1,7,1,7,7) * COS(PHI(1,1,3,2,3,1,7,1,7,7)) +
     4 SMALA(0.5,5,1,6,2,3,3,1,7,1,7,7) * COS(PHI(
     5 3,2,1,3,1,7,1,7,7,7)) + SMALA(0.5,5,1,7,2,3,
     6 7,1,7,1,7,7) * COS(PHI(2,1,3,2,1,7,1,7,7,7))
      GF= - SMALA(C.25,5,1,6,2,3,3,1,7,1,7,7) * COS(PHI
     1 (1,1,1,2,1,3,1,7,7,7)) + SMALA(0.25,5,1,7,2,3,7,1,7,1,
     2 7,7) * COS(PHI(2,1,1,2,1,7,1,7,7,7))-SMALA(0.25,5,
     3 1,5,2,3,3,1,4,1,7,7) * COS(PHI(1,2,1,3,1,4,1,7,7,7))
     4 - SMALA(0.0625,5,1,5,2,4,4,1,7,1,7,7) * COS(PHI(
     5 2,2,1,4,1,7,1,7,7,7)) + SMALA(0.25,5,1,6,2,3,3,1,7,1,7,7)
     6 * COS(PHI(-1,1,1,2,1,3,1,7,7,7)) + SMALA(0.125,5,1,7,
     7 2,3,7,1,7,1,7,7) * COS(PHI(2,1,3,2,1,7,1,7,7,7))
      GII=GD+GE+GF
      SCRIPB(J)=SQRT(GI**2 + GII**2)
      SCRIPB(J)=COS(THET(1))*SCRIPB(J)
      MAXA(J)=(0.5 * AMPF)/(DELT(1)*DELT(2))
      QB(J)=SCRIPB(J)/MAXA(J)
   13 CCNTINUE
      WRITE(6,75)
   75 FORMAT(1H1,4X,1HJ,8X,5HWD(J),5X,9HSCRIPB(J),7X,7HMAXA(J),9X,
     1 5HQB(J),/)
      DO 19 J=1,81
      WRITE(6,85) J,WD(J),SCRIPB(J),MAXA(J),QB(J)
   85 FORMAT(I6,4F20.8)
   19 CONTINUE
      READ(5,92) ITITLE(5),ITITLE(9)
   92 FORMAT(2A8)
```

```
      REAL*8 BCURVE/8H    SCND/
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE THIRD HARMONIC.
C
C
      DO 21 L=1,81
      WD(L)=-5.0+(L-1)*0.125
      DO 23 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(L)*WR)-(WD(L)**2)
      B=DELT(K) * (WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(L)/WR))**2)
     1 +(DELT(K)**2))
      MACH = (2.*AC)/(X*PI*WR*CC*H(1))
   23 CONTINUE
      FACT(3,L) = (((MACH*BETA)**3)*SQRT(6.))/(DELT(1)**3)
      HI=SMALA(0.5,2,1,2,2,1,3,1,7,1,7,7) * SIN(PHI(
     1 1,2,1,3,1,7,1,7,7,7))-SMALA(0.5,4,1,4,2,1,3,2,4,1,7,7)
     2 * SIN(PHI(1,2,2,3,1,4,1,7,7,7)) - SMALA(0.5,4,1,4,2,2,
     3 3,1,4,1,7,7) * SIN(PHI(2,2,1,3,1,4,1,7,7,7)) - SMALA(
     4 0.5,4,1,4,2,2,3,2,7,1,7,7) * SIN(PHI(2,2,2,3,1,7,1,7,7,7))
     5 - SMALA(0.5,4,1,5,2,2,3,1,7,1,7,7) * SIN(PHI(1,1,2,2,
     6 1,3,1,7,7,7))
      HII=SMALA(0.5,2,1,2,2,1,3,1,7,1,7,7) * COS(PHI(
     1 1,2,1,3,1,7,1,7,7,7))-SMALA(0.5,4,1,4,2,1,3,2,4,1,7,7)
     2 * COS(PHI(1,2,2,3,1,4,1,7,7,7)) - SMALA(0.5,4,1,4,2,2,
     3 3,1,4,1,7,7) * COS(PHI(2,2,1,3,1,4,1,7,7,7)) - SMALA(
     4 0.5,4,1,4,2,2,3,2,7,1,7,7) * COS(PHI(2,2,2,3,1,7,1,7,7,7))
     5 - SMALA(0.5,4,1,5,2,2,3,1,7,1,7,7) * COS(PHI(1,1,2,2,
     6 1,3,1,7,7,7))
      SCRIPC(L) = SQRT(HI**2 + HII**2)
      SCRIPC(L)=COS(THET(1))*SCRIPC(L)
      MAXA(L)=0.5*(AMPF**2)*(1./((DELT(1)**2)*DELT(2)*DELT(3)))
      QC(L)=SCRIPC(L)/MAXA(L)
   21 CONTINUE
      WRITE(6,95)
   95 FORMAT(1H1,4X,1HL,8X,5HWD(L),5X,9HSCRIPC(L),7X,7HMAXA(L),
     1 9X,5HQC(L),/)
      DO 67 II=1,81
      WRITE(6,105) II,WD(II),SCRIPC(II),MAXA(II),QC(II)
  105 FORMAT(I6,4F20.8)
```

```
   67 CONTINUE
      READ(5,93) ITITLE(5), ITITLE(9)
   93 FORMAT(2A8)
      REAL*8 CCURVE/8H    THRD/
C
C
C     THIS PORTICN COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE FCLRTH HARMONIC.
C
C
C
      DC 27 JJ=1,81
      WD(JJ)=-5.0+(JJ-1)*0.125
      DC 29 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(JJ)*WR)-(WD(JJ)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(JJ)/WR))**2)
     1 + (DELT(K)**2))
      MACH = (2.*AC)/(X*PI*WR*CC*H(1))
   29 CONTINUE
      WRITE(6,240)
  240 FORMAT(1H0,4X,1HK,10X,1HA,18X,1HB,16X,7HTHET(K),15X,4HH(K),
     1 18X,4HMACH,/)
      DO 241 K=1,6
      WRITE(6,245) K,A,B,THET(K),H(K),MACH
  245 FORMAT(I5,5F20.7)
  241 CCNTINUE
      RA=SMALA(0.5,3,1,3,2,1,3,1,4,1,7,7)* SIN(PHI(
     1 1,2,1,3,1,4,1,7,7,7)) + SMALA(0.125,3,1,3,2,2,4,1,7,1,7,7)
     2 * SIN(PHI(2,2,1,4,1,7,1,7,7,7)) - SMALA(0.25,5,1,5,2,1,
     3 3,1,4,2,5,7) * SIN(PHI(1,2,1,3,2,4,1,5,7,7))-SMALA(
     4 0.0625,5,1,5,2,2,4,2,5,1,7,7) * SIN(PHI(2,2,2,4,1,5,1,7,
     5 7,7)) - SMALA(0.125,5,1,5,2,2,3,1,4,1,5,7) * SIN(PHI(
     6 2,2,1,3,1,4,1,5,7,7)) - SMALA(0.25,5,1,5,2,1,3,2,4,2,7,7)
     7 * SIN(PHI(1,2,2,3,2,4,1,7,7,7))
      RB=-SMALA(0.0625,5,1,5,2,2,3,1,4,1,7,7) * SIN(PHI(
     1 2,2,1,3,2,4,1,7,7,7)) - SMALA(0.25,5,1,5,2,2,3,2,4,1,7,7)
     2 * SIN(PHI(2,2,2,3,1,4,1,7,7,7)) - SMALA(1.,5,1,6,2,2,
     3 3,1,4,1,7,7) * SIN(PHI(1,1,2,2,1,3,1,4,7,7)) - SMALA(
     4 0.125,5,1,5,2,3,3,1,4,1,7,7) * SIN(PHI(3,2,1,3,1,4,1,7,
     5 7,7)) - SMALA(0.125,5,1,6,2,3,4,1,7,1,7,7) * SIN(PHI
     6 (1,1,3,2,1,4,1,7,7,7))
      RI=RA+RB
      RC=SMALA(0.5,3,1,3,2,1,3,1,4,1,7,7) * COS(PHI(
     1 1,2,1,3,1,4,1,7,7,7)) + SMALA(0.125,3,1,3,2,2,4,1,7,1,7,7)
     2 * COS(PHI(2,2,1,4,1,7,1,7,7,7)) - SMALA(0.25,5,1,5,2,1,
     3 3,1,4,2,5,7) * COS(PHI(1,2,1,3,2,4,1,5,7,7)) - SMALA(
     4 0.0625,5,1,5,2,2,4,2,5,1,7,7) * COS(PHI(2,2,2,4,1,5,1,7,
     5 7,7)) - SMALA(0.125,5,1,5,2,2,3,1,4,1,5,7) * COS(PHI(
     6 2,2,1,3,1,4,1,5,7,7)) - SMALA(0.25,5,1,5,2,1,3,2,4,2,7,7)
     7 * COS(PHI(1,2,2,3,2,4,1,7,7,7))
      RD=-SMALA(0.0625,5,1,5,2,2,3,1,4,1,7,7) * COS(PHI(
     1 2,2,1,3,2,4,1,7,7,7)) - SMALA(0.25,5,1,5,2,2,3,2,4,1,7,7)
     2 * COS(PHI(2,2,2,3,1,4,1,7,7,7)) - SMALA(1.,5,1,6,2,2,
     3 3,1,4,1,7,7) * COS(PHI(1,1,2,2,1,3,1,4,7,7)) - SMALA(
     4 0.125,5,1,5,2,3,3,1,4,1,7,7) * COS(PHI(3,2,1,3,1,4,1,7,
     5 7,7)) - SMALA(0.125,5,1,6,2,3,4,1,7,1,7,7) * COS(PHI
     6 (1,1,3,2,1,4,1,7,7,7))
      RII=RC + RC
      SCRIPD(JJ)=(COS(THET(1)))*(SQRT((RI**2)+(RII**2)))
      MAXA(JJ)=0.5*(AMPF**3)*(1./((DELT(1)**3)*DELT(2)*DELT(3)*
     1 DELT(4)))
      QD(JJ)=SCRIPD(JJ)/MAXA(JJ)
   27 CONTINUE
      WRITE(6,115)
  115 FORMAT(1H1,4X,1HK,8X,5HWD(K),5X,9HSCRIPD(K),7X,7HMAXA(K),
     1 9X,5HQD(K),/)
      DO 31 K=1,81
      WRITE(6,125) K,WD(K),SCRIPD(K),MAXA(K),QD(K)
  125 FORMAT(I6,4F20.15)
   31 CONTINUE
      READ(5,94) ITITLE(5),ITITLE(9)
   94 FORMAT(2A8)
      REAL*8 DCURVE/8H    FOTH/
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE FIFTH HARMONIC
C
C
      DO 33 IJ=1,81
      WD(IJ)=-5.0+(IJ-1)*0.125
      DO 37 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(IJ)*WR)-(WD(IJ)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(IJ)/WR))**2)+(DELT(K)**2))
      MACH = (2.*AC)/(X*PI*WR*CC*H(1))
   37 CCNTINUE
      FACT(5,IJ)=(((MACH*BETA)**5)*SQRT(120.))/(DELT(1)**5)
      SI=SMALA(0.5,4,1,4,2,1,3,1,4,1,5,7) * SIN(PHI(
     1 1,2,1,3,1,4,1,5,7,7)) + SMALA(0.125,4,1,4,2,2,4,1,5,1,7,7)
     2 * SIN(PHI(2,2,1,4,1,5,1,7,7,7)) + SMALA(0.25,4,1,4,2,2,
     3 3,1,5,1,7,7) * SIN(PHI(2,2,1,3,1,5,1,7,7,7))
      SII=SMALA(0.5,4,1,4,2,1,3,1,4,1,5,7) * COS(PHI(
     1 1,2,1,3,1,4,1,5,7,7)) + SMALA(0.125,4,1,4,2,2,4,1,5,1,7,7)
     2 * COS(PHI(2,2,1,4,1,5,1,7,7,7))+SMALA(0.25,4,1,4,2,2,
     3 3,1,5,1,7,7) * COS(PHI(2,2,1,3,1,5,1,7,7,7))
      SCRIPE(IJ)=SCRT(SI**2 + SII**2)
      SCRIPE(IJ)=CCS(THET(1))*SCRIPE(IJ)
      MAXA(IJ)=0.5*(AMPF**4)*(1./((DELT(1)**4)*DELT(2)*
     1 DELT(3)*DELT(4) *DELT(5)))
      QE(IJ)=SCRIPE(IJ)/MAXA(IJ)
   33 CCNTINUE
      WRITE(6,135)
  135 FORMAT(1H1,4X,1HI,8X,5HWD(I),5X,9HSCRIPE(I),7X,7HMAXA(I),
     1 9X,5HQE(I),/)
      DO 39 I=1,81
      WRITE(6,145) I,WD(I),SCRIPE(I),MAXA(I),QE(I)
  145 FORMAT(I6,4F20.8)
   39 CONTINUE
      READ(5,96) ITITLE(5),ITITLE(9)
   96 FCRMAT(2A8)
      REAL*8 ECURVE/8H    FITH/
C
C
C     THIS PORTICN COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE SIXTH HARMONIC.
C
C
      DO 41 KK = 1,81
      WD(KK)=-5.0+(KK-1)*0.125
      DO 43 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(KK)*WR)-(WD(KK)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(KK)/WR))**2)
     1 + (DELT(K)**2))
      MACH = (2.*AC)/(X*PI*WR*CC*H(1))
```

```
 43 CONTINUE
    FACT(6,KK)=(((MACH*BETA)**6)*SQRT(720.))/(DELT(1)**6)
    TI=SMALA(0.5,5,1,5,2,1,3,1,4,1,5,6) * SIN(PHI(
   1 1,2,1,3,1,4,1,5,6,7)) + SMALA(0.125,5,1,5,2,2,4,1,5,1,
   2 6,7) * SIN(PHI(2,2,1,4,1,5,1,6,7,7)) + SMALA(0.25,5,
   3 1,5,2,2,3,1,5,1,6,7) * SIN(PHI(2,2,1,3,1,5,1,6,7,7))
   4 + SMALA(0.25,5,1,5,2,2,3,1,4,1,6,7) * SIN(PHI(
   5 2,2,1,3,1,4,1,6,7,7)) + SMALA(0.0625,5,1,5,2,3,4,1,6,1,
   6 7,7) * SIN(PHI(3,2,1,4,1,6,1,7,7,7)) + SMALA(0.125,5,
   7 1,5,2,2,3,2,6,1,7,7) * SIN(PHI(2,2,2,3,1,6,1,7,7,7))
    TII = SMALA(C.5,5,1,5,2,1,3,1,4,1,5,6) * COS(PHI(
   1 1,2,1,3,1,4,1,5,6,7)) + SMALA(0.125,5,1,5,2,2,4,1,5,1,
   2 6,7) * COS(PHI(2,2,1,4,1,5,1,6,7,7)) + SMALA(0.25,5,
   3 1,5,2,2,3,1,5,1,6,7) * COS(PHI(2,2,1,3,1,5,1,6,7,7))
   4 + SMALA(0.25,5,1,5,2,2,3,1,4,1,6,7) * COS(PHI(
   5 2,2,1,3,1,4,1,6,7,7)) + SMALA(0.0625,5,1,5,2,3,4,1,6,1,
   6 7,7) * COS(PHI(3,2,1,4,1,6,1,7,7,7)) + SMALA(0.125,5,
   7 1,5,2,2,3,2,6,1,7,7) * CGS(PHI(2,2,2,3,1,6,1,7,7,7))
    SCRIPF(KK)=SQRT(TI**2+TII**2)
    SCRIPF(KK)=CCS(THET(1))*SCRIPF(KK)
    MAXA(KK)=0.5*(AMPF**5)*(1./((DELT(1)**5)*DELT(2)*
   1 DELT(3) *DELT(4)*DELT(5)*DELT(6)))
    QF(KK)=SCRIPF(KK)/MAXA(KK)
 41 CONTINUE
    WRITE(6,155)
155 FORMAT(1H1,4X,1HI,8X,5HWD(I),5X,9HSCRIPF(I),7X,7HMAXA(I),
   1 9X,5HQF(L),/)
    DO 47 M=1,81
    WRITE(6,165) M,WD(M),SCRIPF(M),MAXA(M),QF(M)
165 FORMAT(I6,4F20.8)
 47 CONTINUE
    READ(5,97) ITITLE(5),ITITLE(9)
 97 FORMAT(2A8)
    REAL*8 FCURVE/8H    SITH/
    STOP
    END
```

```
      FUNCTION SMALA(D,J,K,L,M,II,JJ,KK,LL,MM,IJ,IK)
      DIMENSION THET(7),H(7)
      COMMON THET,H,AMPF
      H(7)=1.
      SMALA=D*AMPF**J*(1./((H(K)**L)*(H(M)**II) *
     1 (H(JJ)**KK) * (H(LL)**MM) * H(IJ) * H(IK)))
      RETURN
      END
```

```
      FUNCTION PHI(I,J,K,L,M,IJ,IK,IL,IM,II)
      DIMENSION THET(7),H(7)
      COMMON THET,H,AMPF
      THET(7)=0.0
      PHI=I*THET(J)+K*THET(L)+M * THET(IJ)
     1 + IK * THET(IL)+THET(IM)+THET(II)
      WRITE(6,600) PHI
  600 FORMAT(F30.15)
      RETURN
      END
```

B-4: Program QCURC

This program is the Q-curve program based upon the Fourier synthesis approach using only the first six harmonics. The curves are normalized in exactly the same manner as in QCURVES and no new quantities have been defined.

```
C
C
C     THIS PROGRAM IS DESIGNED TO COMPUTE AND GRAPH THE
C     Q CURVE FOR EACH HARMONIC USING ONLY THE FIRST TERM OF EACH
C     HARMONIC.  EACH CURVE IS NORMALIZED TO APPROXIMATELY ONE.  THIS
C     IS ACCOMPLISHED BY DIVIDING THE COMPUTED MAGNITUDE OF EACH
C     HARMONIC BY THE MAXIMUM MAGNITUDE OF THAT HARMONIC.
C
      DIMENSION DELT(6),THET(7),H(7),SCRIPA(81),SCRIPB(81),SCRIPC(81),
     1 SCRIPD(81),SCRIPE(81),SCRIPF(81),MAXA(81),WD(81),QA(81),QB(81),
     2 QC(81),QD(81),QE(81),QF(81),FACT(6,81),PHASE(6,81)
      COMMON THET,H,AMPF
      REAL MACH
      REAL MAXA
      REAL*8 ITITLE(12)/'QCURC   ','P. G. RU','FF CURVE',' FOR THE'
     1 ,'        ','HARMONIC','        ','  AO = ','        ',
     2 ,'        ','        ','        '/
      READ(5,10) DELT(1),WR,BETA,X,AO,F
   10 FORMAT(6F10.5)
      WRITE(6,15)
   15 FORMAT(1H1,2X,7HDELT(1),5X,2HWR,7X,4HBETA,7X,1HX,8X,2HAO,9X,
     1 1HF,/)
      WRITE(6,25) DELT(1),WR,BETA,X,AO,F
   25 FORMAT(6F10.5)
      PI=3.1416
      CO = 343.4
      C=PI/F
      AMPF=(AO*BETA)/(X*PI*WR*CO)
      DO 17 I=1,6
      TEMP=I
      DELT(I)=DELT(1)/SQRT(TEMP)
   17 CONTINUE
      WRITE(6,175)
  175 FORMAT(1H0,10X,1HI,10X,1HC,10X,4HAMPF,10X,7HDELT(I),/)
      DO 57 I=1,6
      WRITE(6,185) I,C,AMPF,DELT(I)
  185 FORMAT(I11,F16.5,F13.10,F15.7)
   57 CONTINUE
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE FIRST HARMONIC
```

```
C
C
      DO 9 M=1,81
      WD(M)=-5.0+(M-1)*0.125
      DO 3 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(M)*WR)-(WD(M)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(M)/WR))**2)+(DELT(K)**2))
      MACH=(2.*AO)/(X*PI*WR*CO*H(1))
    3 CONTINUE
      FACT(1,M)=(((MACH*BETA)**1)*SQRT(1.))/(DELT(1)**1)
      WRITE(6,230)
  230 FORMAT(1HO,4X,1HK,10X,1HA,18X,1HB,16X,7HTHET(K),15X,4HH(K),
     1 18X,4HMACH,/)
      DO 211 K=1,6
      WRITE(6,220) K,A,B,THET(K),H(K),MACH
  220 FORMAT(I5,5F20.7)
  211 CONTINUE
      SCRIPA(M) = COS(THET(1))
      MAXA(M)=1.
      QA(M)=SCRIPA(M)/MAXA(M)
    9 CONTINUE
      WRITE(6,55)
   55 FORMAT(1H1,4X,1HM,8X,5HWD(M),5X,9HSCRIPA(M),7X,7HMAXA(M),9X,
     1 5HQA(M),/)
      DO 11 M=1,81
      WRITE(6,65) M,WD(M),SCRIPA(M),MAXA(M),QA(M)
   65 FORMAT(I6,4F20.8)
   11 CONTINUE
      READ(5,91) ITITLE(5),ITITLE(9)
   91 FORMAT(2A8)
      REAL*8 ACURVE/8H    FRST/
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE SECOND HARMONIC.
C
C
      DO 13 J=1,81
      WD(J)=-5.0+(J-1)*0.125
      DO 59 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(J)*WR)-(WD(J)**2)
      B=DELT(K) * (WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(J)/WR))**2)
     1 +(DELT(K)**2))
      MACH = (2.*AC)/(X*PI*WR*CC*H(1))
   59 CONTINUE
      FACT(2,J)=(((MACH*BETA)**2)*SQRT(2.))/(DELT(1)**2)
      GI=SMALA(0.5,1,1,1,2,1,7,1,7,1,7,7) * SIN(PHI
     1 (1,2,1,7,1,7,1,7,7,7))
      GD=SMALA(0.5,1,1,1,2,1,7,1,7,1,7,7) * COS(PHI(
     1 1,2,1,7,1,7,1,7,7,7))
      GII = GD
      SCRIPB(J)=(COS(THET(1)))*(SQRT((GI**2)+(GII**2)))
      MAXA(J)=(0.5 * AMPF)/(DELT(1)*DELT(2))
      QB(J)=SCRIPB(J)/MAXA(J)
   13 CONTINUE
      WRITE(6,75)
   75 FORMAT(1H1,4X,1HJ,8X,5HWD(J),5X,9HSCRIPB(J),7X,7HMAXA(J),9X,
     1 5HQB(J),/)
      DO 19 J=1,81
      WRITE(6,85) J,WD(J),SCRIPB(J),MAXA(J),QB(J)
   85 FORMAT(I6,4F20.8)
   19 CONTINUE
      READ(5,92) ITITLE(5),ITITLE(9)
   92 FORMAT(2A8)
      REAL*8 BCURVE/8H    SCND/
      CALL DRAW(81,WD,QB,0,0,BCURVE,ITITLE,2.50E+00,0,0,0,
     1 0,1,6,8,0,L)
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE THIRD HARMONIC.
C
C
      DO 21 L=1,81
      WD(L)=-5.0+(L-1)*0.125
      DO 23 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(L)*WR)-(WD(L)**2)
      B=DELT(K) * (WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(L)/WR))**2)
     1 +(DELT(K)**2))
      MACH = (2.*AC)/(X*PI*WR*CC*H(1))
   23 CONTINUE
      FACT(3,L) = (((MACH*BETA)**3)*SQRT(6.))/(DELT(1)**3)
      HI=SMALA(0.5,2,1,2,2,1,3,1,7,1,7,7) * SIN(PHI(
     1 1,2,1,3,1,7,1,7,7,7))
      HII=SMALA(0.5,2,1,2,2,1,3,1,7,1,7,7) * COS(PHI(
     1 1,2,1,3,1,7,1,7,7,7))
      SCRIPC(L)=(CCS(THET(1)))*(SQRT((HI**2)+(HII**2)))
      MAXA(L)=0.5*(AMPF**2)*(1./((DELT(1)**2)*DELT(2)*DELT(3)))
      QC(L)=SCRIPC(L)/MAXA(L)
   21 CONTINUE
      WRITE(6,95)
   95 FORMAT(1H1,4X,1HL,8X,5HWD(L),5X,9HSCRIPC(L),7X,7HMAXA(L),
     1 9X,5HQC(L),/)
      DO 67 II=1,81
      WRITE(6,105) II,WD(II),SCRIPC(II),MAXA(II),QC(II)
  105 FORMAT(I6,4F20.8)
   67 CONTINUE
      READ(5,93) ITITLE(5), ITITLE(9)
   93 FORMAT(2A8)
      REAL*8 CCURVE/8H    THRD/
      CALL DRAW(81,WD,QC,0,0,CCURVE,ITITLE,2.50E+0C,0,0,0,
     1 0,1,6,8,0,L)
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE FOURTH HARMONIC.
C
C
      DO 27 JJ=1,81
      WD(JJ)=-5.0+(JJ-1)*0.125
      DO 29 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(JJ)*WR)-(WD(JJ)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(JJ)/WR))**2)
     1 + (DELT(K)**2))
      MACH = (2.*AC)/(X*PI*WR*CO*H(1))
   29 CONTINUE
      WRITE(6,240)
  240 FORMAT(1H0,4X,1HK,10X,1HA,18X,1HB,16X,7HTHET(K),15X,4HH(K),
     1 18X,4HMACH,/)
      DC 241 K=1,6
      WRITE(6,245) K,A,B,THET(K),H(K),MACH
```

```
245 FORMAT(I5,5F20.7)
241 CONTINUE
    RI=SMALA(0.5,3,1,3,2,1,3,1,4,1,7,7)* SIN(PHI(
   1 1,2,1,3,1,4,1,7,7,7)) + SMALA(0.125,3,1,3,2,2,4,1,7,1,7,7)
   2 * SIN(PHI(2,2,1,4,1,7,1,7,7,7))
    RC=SMALA(0.5,3,1,3,2,1,3,1,4;1,7,7) * COS(PHI(
   1 1,2,1,3,1,4,1,7,7,7)) + SMALA(0.125,3,1,3,2,2,4,1,7,1,7,7)
   2 * COS(PHI(2,2,1,4,1,7,1,7,7,7))
    RII = RC
    SCRIPD(JJ)=(COS(THET(1)))*(SQRT((RI**2)+(RII**2)))
    MAXA(JJ)=0.5*(AMPF**3)*(1./((DELT(1)**3)*DELT(2)*DELT(3)*
   1 DELT(4)))
    QD(JJ)=SCRIPC(JJ)/MAXA(JJ)
 27 CONTINUE
    WRITE(6,115)
115 FORMAT(1H1,4X,1HK,8X,5HWD(K),5X,9HSCRIPD(K),7X,7HMAXA(K),
   1 9X,5HQD(K),/)
    DO 31 K=1,81
    WRITE(6,125) K,WD(K),SCRIPD(K),MAXA(K),QD(K)
125 FORMAT(I6,4F20.15)
 31 CONTINUE
    READ(5,94) ITITLE(5),ITITLE(9)
 94 FORMAT(2A8)
    REAL*8 DCURVE/8H    FOTH/
    CALL DRAW(81,WC,QD,0,0,DCURVE,ITITLE,2.50E+00,0,0,0,
   1 0,1,6,8,0,L)
C
C
C     THIS PORTICN COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE FIFTH HARMONIC
C
C
    DO 33 IJ=1,81
    WD(IJ)=-5.0+(IJ-1)*C.125
    DO 37 K=1,6
    A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(IJ)*WR)-(WD(IJ)**2)
    B=DELT(K)*(WR**2)
    THET(K)=ATAN2(A,B)
    H(K)=SQRT(((CELT(1)-DELT(K)-(2.*WD(IJ)/WR))**2)+(DELT(K)**2))
    MACH = (2.*AC)/(X*PI*WR*CO*H(1))
 37 CONTINUE
    SI=SMALA(0.5,4,1,4,2,1,3,1,4,1,5,7) * SIN(PHI(
   1 1,2,1,3,1,4,1,5,7,7)) + SMALA(0.125,4,1,4,2,2,4,1,5,1,7,7)
```

```
     2 * SIN(PHI(2,2,1,4,1,5,1,7,7,7)) + SMALA(0.25,4,1,4,2,2,
     3 3,1,5,1,7,7) * SIN(PHI(2,2,1,3,1,5,1,7,7,7))
      SII=SMALA(0.5,4,1,4,2,1,3,1,4,1,5,7) * COS(PHI(
     1 1,2,1,3,1,4,1,5,7,7)) + SMALA(0.125,4,1,4,2,2,4,1,5,1,7,7)
     2 * COS(PHI(2,2,1,4,1,5,1,7,7,7))+SMALA(0.25,4,1,4,2,2,
     3 3,1,5,1,7,7) * COS(PHI(2,2,1,3,1,5,1,7,7,7))
      SCRIPE(IJ)=(COS(THET(1)))*(SQRT((SI**2)+(SII**2)))
      MAXA(IJ)=0.5*(AMPF**4)*(1./((DELT(1)**4)*DELT( 2)*
     1 DELT(3)*DELT(4)  *DELT(5)))
      QE(IJ)=SCRIPE(IJ)/MAXA(IJ)
   33 CONTINUE
      WRITE(6,135)
  135 FORMAT(1H1,4X,1HI,8X,5HWD(I),5X,9HSCRIPE(I),7X,7HMAXA(I),
     1 9X,5HQE(I),/)
      DO 39 I=1,81
      WRITE(6,145) I,WD(I),SCRIPE(I),MAXA(I),QE(I)
  145 FORMAT(I6,4F20.8)
   39 CONTINUE
      READ(5,96) ITITLE(5),ITITLE(9)
   96 FORMAT(2A8)
      REAL*8 ECURVE/8H    FITH/
      CALL DRAW(81,WD,QE,0,0,ECURVE,ITITLE,2.50E+00,0,0,0,
     1 0,1,6,8,0,L)
C
C
C     THIS PORTICN COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE SIXTH HARMONIC.
C
C

      DO 41 KK = 1,81
      WD(KK)=-5.0+(KK-1)*0.125
      DO 43 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(KK)*WR)-(WD(KK)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(KK)/WR))**2)
     1 + (DELT(K)**2))
      MACH = (2.*AC)/(X*PI*WR*CO*H(1))
   43 CONTINUE
      FACT(6,KK)=(((MACH*BETA)**6)*SQRT(720.))/(DELT(1)**6)
      TI=SMALA(0.5,5,1,5,2,1,3,1,4,1,5,6) * SIN(PHI(
     1 1,2,1,3,1,4,1,5,6,7)) + SMALA(0.125,5,1,5,2,2,4,1,5,1,
     2 6,7) * SIN(PHI(2,2,1,4,1,5,1,6,7,7)) + SMALA(0.25,5,
     3 1,5,2,2,3,1,5,1,6,7) * SIN(PHI(2,2,1,3,1,5,1,6,7,7))
     4 + SMALA(0.25,5,1,5,2,2,3,1,4,1,6,7) * SIN(PHI(
     5 2,2,1,3,1,4,1,6,7,7)) + SMALA(0.0625,5,1,5,2,3,4,1,6,1,
     6 7,7) * SIN(PHI(3,2,1,4,1,6,1,7,7,7)) + SMALA(0.125,5,
     7 1,5,2,2,3,2,6,1,7,7) * SIN(PHI(2,2,2,3,1,6,1,7,7,7))
      TII = SMALA(C.5,5,1,5,2,1,3,1,4,1,5,6) * COS(PHI(
     1 1,2,1,3,1,4,1,5,6,7)) + SMALA(0.125,5,1,5,2,2,4,1,5,1,
     2 6,7) * COS(PHI(2,2,1,4,1,5,1,6,7,7)) + SMALA(0.25,5,
     3 1,5,2,2,3,1,5,1,6,7) * COS(PHI(2,2,1,3,1,5,1,6,7,7))
     4 + SMALA(0.25,5,1,5,2,2,3,1,4,1,6,7) * COS(PHI(
     5 2,2,1,3,1,4,1,6,7,7)) + SMALA(0.0625,5,1,5,2,3,4,1,6,1,
     6 7,7) * COS(PHI(3,2,1,4,1,6,1,7,7,7)) + SMALA(0.125,5,
     7 1,5,2,2,3,2,6,1,7,7) * COS(PHI(2,2,2,3,1,6,1,7,7,7))
      SCRIPF(KK)=(COS(THET(1)))*(SQRT((TI**2)+(TII**2)))
      MAXA(KK)=0.5*(AMPF**5)*(1./((DELT(1)**5)*DELT(2)*
     1 DELT(3) *DELT(4)*DELT(5)*DELT(6)))
      QF(KK)=SCRIPF(KK)/MAXA(KK)
   41 CONTINUE
      WRITE(6,155)
  155 FORMAT(1H1,4X,1HI,8X,5HWD(I),5X,9HSCRIPF(I),7X,7HMAXA(I),
     1 9X,5HQF(L),/)
      DO 47 M=1,81
      WRITE(6,165) M,WD(M),SCRIPF(M),MAXA(M),QF(M)
  165 FORMAT(I6,4F20.8)
   47 CONTINUE
      READ(5,97) ITITLE(5),ITITLE(9)
   97 FORMAT(2A8)
      REAL*8 FCURVE/8H    SITH/
      CALL DRAW(81,WD,QF,0,0,FCURVE,ITITLE,2.50E+00,0,0,0,
     1 0,1,6,8,0,L)
      STOP
      END
      FUNCTION SMALA(D,J,K,L,M,II,JJ,KK,LL,MM,IJ,IK)
      COMMON THET,H,AMPF
      DIMENSICN THET(7),H(7)
      H(7)=1.
      SMALA=D*AMPF**J*(1./((H(K)**L)*(H(M)**II) *
     1 (H(JJ)**KK) * (H(LL)**MM) * H(IJ) * H(IK)))
      RETURN
      END
```

```
      FUNCTION PHI(I,J,K,L,M,IJ,IK,IL,IM,II)
      DIMENSION THET(7),H(7)
      COMMON THET,H,AMFF
      THET(7)=0.0
      PHI=I*THET(J)+K*THET(L)+M * THET(IJ)
     1 + IK * THET(IL)+THET(IM)+THET(II)
      WRITE(6,600) PHI
  600 FORMAT(F30.15)
      RETURN
      END
```

B-5: Program PHAMP

This program was designed to compute the amplitudes and associated phase angles for each harmonic. Essentially this program is the QCURVES program with only minor modifications. One new quantity has been defined as PHASE(I,J). This is the phase angle associated with the i-th harmonic and the j-th $\Delta\omega$. In all other respects Table B.2 is applicable.

```
C
C
C     THIS PROGRAM IS DESIGNED TO COMPUTE THE AMPLITUDE AND PHASE
C     OF EACH HARMONIC AS THE FREQUENCY IS SWEPT THROUGH RESONANCE.
C     EACH AMPLITUDE IS PRINTED OUT IN BOTH A NORMALIZED AND AN
C     UN-NORMALIZED FORM. THE NORMALIZED FORM HAS BEEN NORMALIZED BY
C     THE AMPLITUDE OF THE FIRST TERM OF THAT HARMONIC.
C
C
      DIMENSION DELT(6),THET(7),H(7),SCRIPA(41),SCRIPB(41),SCRIPC(41),
     1 SCRIPD(41),SCRIPE(41),SCRIPF(41),MAXA(41),WD(41),QA(41),QB(41),
     2 QC(41),QD(41),QE(41),QF(41),FACT(6,41),PHASE(6,41)
      COMMON THET,H,AMPF
      REAL MACH
      REAL MAXA
      READ(5,10) DELT(1),WR,BETA,X,A0,F
   10 FORMAT(6F10.5)
      WRITE(6,15)
   15 FORMAT(1H1,2X,7HDELT(1),5X,2HWR,7X,4HBETA,7X,1HX,8X,2HA0,9X,
     1 1HF,/)
      WRITE(6,25) DELT(1),WR,BETA,X,A0,F
   25 FORMAT(6F10.5)
      PI=3.1416
      C0 = 343.4
      C=PI/F
      AMPF=(A0*BETA)/(X*PI*WR*C0)
      DO 17 I=1,6
      TEMP=I
      DELT(I)=DELT(1)/SQRT(TEMP)
   17 CONTINUE
      WRITE(6,175)
  175 FORMAT(1H0,10X,1HI,10X,1HC,10X,4HAMPF,10X,7HDELT(I),/)
      DO 57 I=1,6
      WRITE(6,185) I,C,AMPF,DELT(I)
  185 FORMAT(I11,F16.5,F13.10,F15.7)
   57 CONTINUE
C
C
C     THIS PORTION OF THE PROGRAM COMPUTES THE AMPLITUDE AND PHASE OF
C     THE FIRST HARMONIC
C
C
      DO 9 M=1,41
      WD(M)=-5.0+(M-1)*0.25
      DO 3 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(M)*WR)-(WD(M)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(M)/WR))**2)+(DELT(K)**2))
      MACH=(2.*A0)/(X*PI*WR*CO*H(1))
    3 CONTINUE
      FACT(1,M)=(((MACH*BETA)**1)*SQRT(1.))/(DELT(1)**1)
      FI = SMALA(-0.5,2,1,3,2,1,7,1,7,1,7,7) * SIN(
     1 PHI(1,1,1,2,1,7,1,7,7,7))+SMALA(0.5,4,1,5,2,2,3,1,7,1,7,7)*SIN
     2  (PHI(1,1,2,2,1,3,1,7,7,7))+ SMALA(-0.25,4,1,5,2,2,3,1,7,
     3 1,7,7) * SIN(PHI(1,1,1,3,1,7,1,7,7,7))
     4 + SMALA(0.5,4,1,6,2,2,7,1,7,1,7,7)*SIN(
     5 PHI(2,1,1,2,1,7,1,7,7,7))+ SMALA(0.25,4,1,6,2,2,7,1,7,1,7,7)*SIN
     6  (PHI(2,1,1,7,1,7,1,7,7,7))
      FII=1+SMALA(-0.5,2,1,3,2,1,7,1,7,1,7,7)*
     1 COS(PHI(1,1,1,2,1,7,1,7,7,7))+ SMALA(0.5,4,1,5,2,2,
     2 3,1,7,1,7,7)*COS(PHI(1,1,2,2,1,3,1,7,7,7)) +
     3 SMALA(-0.25,4,1,5,2,2,3,1,7,1,7,7)* COS(PHI(
     4 1,1,1,3,1,7,1,7,7,7))+ SMALA(0.5,4,1,6,2,2,7,1,7,1,7,7)
     5 * COS(PHI(2,1,1,2,1,7,1,7,7,7)) + SMALA(0.25,4,
     6 1,6,2,2,7,1,7,1,7,7) * COS(PHI(2,1,1,7,1,7,1,7,7,7))
      WRITE(6,210)
  210 FORMAT(1H0,5X,1HM,10X,2HFI,18X,3HFII,17X,9HFACT(1,M),/)
```

```
      WRITE(6,200) M,FI,FII,FACT(1,M)
  200 FORMAT(I6,3F20.5)
      SCRIPA(M)=SQRT(FI**2+FII**2)
      SCRIPA(M)=COS(THET(1))*SCRIPA(M)
      MAXA(M)=1.
      QA(M)=SCRIPA(M)/MAXA(M)
      A=FI
      B=FII
      PHASE(1,M)=ATAN2(A,B)
    9 CONTINUF
      WRITE(6,55)
   55 FORMAT(1H1,4X,1HM,8X,5HWD(M),5X,9HSCRIPA(M),7X,7HMAXA(M),9X,
     1 5HQA(M),10X,10HPHASE(1,M),/)
      DO 11 M=1,41
      WRITE(6,65) M,WD(M),SCRIPA(M),MAXA(M),QA(M),PHASE(1,M)
   65 FORMAT(I6,5F20.8)
   11 CONTINUE
C
C
C     THIS PORTION OF THE PROGRAM COMPUTES THE AMPLITUDE AND PHASE OF
C     THE SECOND HARMONIC
C
C
      DO 13 J=1,41
      WD(J)=-5.0+(J-1)*0.25
      DO 59 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(J)*WR)-(WD(J)**2)
      B=DELT(K) * (WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(J)/WR))**2)
     1 +(DELT(K)**2))
      MACH = (2.*AO)/(X*PI*WR*CO*H(1))
   59 CONTINUE
      FACT(2,J)=(((MACH*BETA)**2)*SQRT(2.))/(DELT(1)**2)
```

```
 GA=SMALA(0.5,1,1,1,2,1,7,1,7,1,7,7) * SIN(PHI
1 (1,2,1,7,1,7,1,7,7,7))-SMALA(0.5,3,1,3,2,2,3,1,7,1,7,7)
2 * SIN(PHI(2,2,1,3,1,7,1,7,7,7)) -SMALA(0.5,3,1,4,
3 2,2,7,1,7,1,7,7)* SIN(PHI(1,1,2,2,1,7,1,7,7,7)) +
4 SMALA(0.5,5,1,5,2,2,3,2,4,1,7,7) * SIN(PHI(2,2,2,3,1,4,
5 1,7,7,7)) + SMALA(0.125,5,1,5,2,3,3,1,4,1,7,7)*
6 SIN(PHI(3,2,1,3,1,4,1,7,7,7))
 GB = SMALA(0.5,5,1,5,2,3,3,2,7,1,7,7) * SIN(PHI
1 (3,2,2,3,1,7,1,7,7,7))+ SMALA(0.5,5,1,6,2,3,3,1,
2 7,1,7,7) * SIN(PHI(1,1,3,2,1,3,1,7,7,7)) +
3 SMALA(0.25,5,1,6,2,3,3,1,7,1,7,7) * SIN(PHI(1,1,
4 3,2,3,1,7,1,7,7)) + SMALA(0.50,5,1,6,2,3,3,1,7,1,
5 7,7) * SIN(PHI(3,2,1,3,1,7,1,7,7,7)) + SMALA(0.5,5,
6 1,7,2,3,7,1,7,1,7,7) * SIN(PHI(2,1,3,2,1,7,1,7,7,7))
 GC=-SMALA(0.25,5,1,6,2,3,3,1,7,1,7,7) * SIN(PHI(
1 1,1,1,2,1,3,1,7,7,7)) + SMALA(0.25,5,1,7,2,3,7,1,7,1,
2 7,7) * SIN(PHI(2,1,1,2,1,7,1,7,7,7)) - SMALA(0.25,5,
3 1,5,2,3,3,1,4,1,7,7) * SIN(PHI(1,2,1,3,1,4,1,7,7,7))
4 - SMALA(0.0625,5,1,5,2,4,4,1,7,1,7,7) * SIN (PHI(
5 2,2,1,4,1,7,1,7,7,7)) + SMALA(0.25,5,1,6,2,3,3,1,7,1,7,7)
6 * SIN(PHI(-1,1,1,2,1,3,1,7,7,7)) + SMALA(0.125,5,1,7,
7 2,3,7,1,7,1,7,7) * SIN(PHI(2,1,3,2,1,7,1,7,7,7))
 GI=GA+GB+GC
 GD=SMALA(0.5,1,1,1,2,1,7,1,7,1,7,7) * COS(PHI(
1 1,2,1,7,1,7,1,7,7,7)) - SMALA(0.5,3,1,3,2,2,3,1,7,1,7,7)
6 * COS(PHI(2,2,1,3,1,7,1,7,7,7)) - SMALA(0.5,3,1,4,
3 2,2,7,1,7,1,7,7)*COS(PHI(1,1,2,2,1,7,1,7,7,7)) +
4 SMALA(0.5,5,1,5,2,2,3,2,4,1,7,7) * COS(PHI(2,2,2,3,1,4,
5 1,7,7,7)) + SMALA(0.125,5,1,5,2,3,3,1,4,1,7,7) *
6 COS(PHI(3,2,1,3,1,4,1,7,7,7))
 GE=SMALA(0.5,5,1,5,2,3,3,2,7,1,7,7) * COS(PHI(
1 3,2,2,3,1,7,1,7,7,7)) + SMALA(0.5,5,1,6,2,3,3,1,
2 7,1,7,7) * COS(PHI(1,1,3,2,1,3,1,7,7,7)) + SMALA(0.25,
3 5,1,6,2,3,3,1,7,1,7,7) * COS(PHI(1,1,3,2,3,1,7,1,7,7)) +
     4 SMALA(0.5,5,1,6,2,3,3,1,7,1,7,7) * COS(PHI(
     5 3,2,1,3,1,7,1,7,7,7)) + SMALA(0.5,5,1,7,2,3,
     6 7,1,7,1,7,7) * COS(PHI(2,1,3,2,1,7,1,7,7,7))
      GF= - SMALA(0.25,5,1,6,2,3,3,1,7,1,7,7) * COS(PHI
     1 (1,1,1,2,1,3,1,7,7,7)) + SMALA(0.25,5,1,7,2,3,7,1,7,1,
     2 7,7) * COS(PHI(2,1,1,2,1,7,1,7,7,7))-SMALA(0.25,5,
     3 1,5,2,3,3,1,4,1,7,7) * COS(PHI(1,2,1,3,1,4,1,7,7,7))
     4 - SMALA(0.0625,5,1,5,2,4,4,1,7,1,7,7) * COS(PHI(
     5 2,2,1,4,1,7,1,7,7,7)) + SMALA(0.25,5,1,6,2,3,3,1,7,1,7,7)
     6 * COS(PHI(-1,1,1,2,1,3,1,7,7,7)) + SMALA(0.125,5,1,7,
     7 2,3,7,1,7,1,7,7) * COS(PHI(2,1,3,2,1,7,1,7,7,7))
      GII=GD+GE+GF
      SCRIPB(J)=SQRT(GI**2 + GII**2)
      SCRIPB(J)=COS(THET(1))*SCRIPB(J)
      MAXA(J)=(0.5 * AMPF)/(DELT(1)*DELT(2))
      QB(J)=SCRIPB(J)/MAXA(J)
      A=GI
      B=GII
      PHASE(2,J)=ATAN2(A,B)
   13 CONTINUE
      WRITE(6,75)
   75 FORMAT(1H1,4X,1HJ,8X,5HWD(J),5X,9HSCRIPB(J),7X,7HMAXA(J),9X,
     1 5HQB(J),10X,10HPHASE(2,J),/)
      DO 19 J=1,41
      WRITE(6,85) J,WD(J),SCRIPB(J),MAXA(J),QB(J),PHASE(2,J)
   85 FORMAT(I6,5F20.8)
   19 CONTINUE
C
C
C     THIS PORTION OF THE PROGRAM COMPUTES THE AMPLITUDE AND PHASE OF
C     THE THIRD HARMONIC.
C
C
      DO 21 L=1,41
      WD(L)=-5.0+(L-1)*0.25
      DO 23 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(L)*WR)-(WD(L)**2)
      B=DELT(K) * (WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(L)/WR))**2)
     1 +(DELT(K)**2))
      MACH = (2.*A0)/(X*PI*WR*C0*H(1))
   23 CONTINUE
      FACT(3,L) = (((MACH*BETA)**3)*SQRT(6.))/(DELT(1)**3)
      WRITE(6,230)
  230 FORMAT(1H1,4X,1HK,10X,1HA,18X,1HB,16X,7HTHET(K),15X,4HH(K),
     1 18X,4HMACH,/)
      DO 211 K=1,6
      WRITE(6,220) K,A,B,THET(K),H(K),MACH
  220 FORMAT(I5,5F20.7)
  211 CONTINUE
      HI=SMALA(0.5,2,1,2,2,1,3,1,7,1,7,7) * SIN(PHI(
     1 1,2,1,3,1,7,1,7,7,7))-SMALA(0.5,4,1,4,2,1,3,2,4,1,7,7)
     2 * SIN(PHI(1,2,2,3,1,4,1,7,7,7)) - SMALA(0.5,4,1,4,2,2,
     3 3,1,4,1,7,7) * SIN(PHI(2,2,1,3,1,4,1,7,7,7)) - SMALA(
     4 0.5,4,1,4,2,2,3,2,7,1,7,7) * SIN(PHI(2,2,2,3,1,7,1,7,7,7))
     5 - SMALA(0.5,4,1,5,2,2,3,1,7,1,7,7) * SIN(PHI(1,1,2,2,
     6 1,3,1,7,7,7))
      HII=SMALA(0.5,2,1,2,2,1,3,1,7,1,7,7) * COS(PHI(
     1 1,2,1,3,1,7,1,7,7,7))-SMALA(0.5,4,1,4,2,1,3,2,4,1,7,7)
     2 * COS(PHI(1,2,2,3,1,4,1,7,7,7)) - SMALA(0.5,4,1,4,2,2,
     3 3,1,4,1,7,7) * COS(PHI(2,2,1,3,1,4,1,7,7,7)) - SMALA(
     4 0.5,4,1,4,2,2,3,2,7,1,7,7) * COS(PHI(2,2,2,3,1,7,1,7,7,7))
     5 - SMALA(0.5,4,1,5,2,2,3,1,7,1,7,7) * COS(PHI(1,1,2,2,
     6 1,3,1,7,7,7))
      SCRIPC(L) = SQRT(HI**2 + HII**2)
      SCRIPC(L)=COS(THET(1))*SCRIPC(L)
      MAXA(L)=0.5*(AMPF**2)*(1./((DELT(1)**2)*DELT(2)*DELT(3)))
      QC(L)=SCRIPC(L)/MAXA(L)
      A=HI
      B=HII
      PHASE(3,L)=ATAN2(A,B)
   21 CONTINUE
      WRITE(6,95)
   95 FORMAT(1H1,4X,1HL,8X.5HWD(L),5X,9HSCRIPC(L),7X,7HMAXA(L),
     1 9X,5HQC(L),10X,10HPHASE(3,L),/)
      DO 67 II=1,41
      WRITE(6,105) II,WD(II),SCRIPC(II),MAXA(II),QC(II),PHASE(3,II)
  105 FORMAT(I6,5F20.3)
   67 CONTINUE
C
C
C     THIS PORTION OF THE PROGRAM COMPUTES THE AMPLITUDE AND PHASE OF
C     THE FOURTH HARMONIC.
C
C
      DO 27 JJ=1,41
      WD(JJ)=-5.0+(JJ-1)*0.25
      DO 29 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(JJ)*WR)-(WD(JJ)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(JJ)/WR))**2)
     1 + (DELT(K)**2))
      MACH = (2.*AO)/(X*PI*WR*CO*H(1))
   29 CONTINUE
      FACT(4,J)=(((MACH*BETA)**4)*SQRT(24.))/(DELT(1)**4)
      RA=SMALA(0.5,3,1,3,2,1,3,1,4,1,7,7)* SIN(PHI(
     1 1,2,1,3,1,4,1,7,7,7)) + SMALA(0.125,3,1,3,2,2,4.1,7,1,7,7)
     2 * SIN(PHI(2,2,1,4,1,7,1,7,7,7)) - SMALA(0.25,5,1,5,2,1,
     3 3,1,4,2,5,7) * SIN(PHI(1,2,1,3,2,4,1,5,7,7))-SMALA(
     4 0.0625,5,1,5,2,2,4,2,5,1,7,7) * SIN(PHI(2,2,2,4,1,5,1,7,
5 7,7)) - SMALA(0.125,5,1,5,2,2,3,1,4,1,5,7) * SIN(PHI(
6 2,2,1,3,1,4,1,5,7,7)) - SMALA(0.25,5,1,5,2,1,3,2,4,2,7,7)
7 * SIN(PHI(1,2,2,3,2,4,1,7,7,7))
 RB=-SMALA(0.0625,5,1,5,2,2,3,1,4,1,7,7) * SIN(PHI(
1 2,2,1,3,2,4,1,7,7,7)) - SMALA(0.25,5,1,5,2,2,3,2,4,1,7,7)
2 * SIN(PHI(2,2,2,3,1,4,1,7,7,7)) - SMALA(1.,5,1,6,2,2,
3 3,1,4,1,7,7) * SIN(PHI(1,1,2,2,1,3,1,4,7,7)) - SMALA(
4 0.125,5,1,5,2,3,3,1,4,1,7,7) * SIN(PHI(3,2,1,3,1,4,1,7,
5 7,7)) - SMALA(0.125,5,1,6,2,3,4,1,7,1,7,7) * SIN(PHI
6 (1,1,3,2,1,4,1,7,7,7))
 RI=RA+RB
 RC=SMALA(0.5,3,1,3,2,1,3,1,4,1,7,7) * COS(PHI(
1 1,2,1,3,1,4,1,7,7,7)) + SMALA(0.125,3,1,3,2,2,4,1,7,1,7,7)
2 * COS(PHI(2,2,1,4,1,7,1,7,7,7)) - SMALA(0.25,5,1,5,2,1,
3 3,1,4,2,5,7) * COS(PHI(1,2,1,3,2,4,1,5,7,7)) - SMALA(
4 0.0625,5,1,5,2,2,4,2,5,1,7,7) * COS(PHI(2,2,2,4,1,5,1,7,
5 7,7)) - SMALA(0.125,5,1,5,2,2,3,1,4,1,5,7) * COS(PHI(
6 2,2,1,3,1,4,1,5,7,7)) - SMALA(0.25,5,1,5,2,1,3,2.4,2,7,7)
7 * COS(PHI(1,2,2,3,2,4,1,7,7,7))
 RD=-SMALA(0.0625,5,1,5,2,2,3,1,4,1,7,7) * COS(PHI(
1 2,2,1,3,2,4,1,7,7,7)) - SMALA(0.25,5,1,5,2,2,3,2,4,1,7,7)
2 * COS(PHI(2,2,2,3,1,4,1,7,7,7)) - SMALA(1.,5,1,6,2,2,
3 3,1,4,1,7,7) * COS(PHI(1,1,2,2,1,3,1,4,7,7)) - SMALA(
4 0.125,5,1,5,2,3,3,1,4,1,7,7) * COS(PHI(3,2,1,3,1,4,1,7,
5 7,7)) - SMALA(0.125,5,1,6,2,3,4,1,7,1,7,7) * COS(PHI
6 (1,1,3,2,1,4,1,7,7,7))
 RII=RC + RD
 SCRIPD(JJ)=SQRT(RI**2+RII**2)
 SCRIPD(JJ)=COS(THET(1))*SCRIPD(JJ)
 MAXA(JJ)=0.5*(AMPF**3)*(1./((DELT(1)**3)*DELT(2)*DELT(3)*
1 DELT(4)))
 QD(JJ)=SCRIPD(JJ)/MAXA(JJ)
 A=RI
 B=RII
```

```
      PHASE(4,JJ) = ATAN2(A,B)
   27 CONTINUE
      WRITE(6,115)
  115 FORMAT(1H1,4X,1HK,8X,5HWD(K),5X,9HSCRIPD(K),7X,7HMAXA(K),
     1 9X,5HQD(K),10X,10HPHASE(4,K),/)
      DO 31 K=1,41
      WRITE(6,125) K,WD(K),SCRIPD(K),MAXA(K),QD(K),PHASE(4,K)
  125 FORMAT(I6,5F20.8)
   31 CONTINUE
C
C
C     THIS PORTION OF THE PROGRAM COMPUTES THE AMPLITUDE AND PHASE OF
C     THE FIFTH HARMONIC.
C
C
      DO 33 IJ=1,41
      WD(IJ)=-5.0+(IJ-1)*0.25
      DO 37 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(IJ)*WR)-(WD(IJ)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(IJ)/WR))**2)+(DELT(K)**2))
      MACH = (2.*A0)/(X*PI*WR*C0*H(1))
   37 CONTINUE
      FACT(5,IJ)=(((MACH*BETA)**5)*SQRT(120.))/(DELT(1)**5)
      WRITE(6,240)
  240 FORMAT(1H0,4X,1HK,10X,1HA,18X,1HB,16X,7HTHET(K),15X,4HH(K),
     1 18X,4HMACH,/)
      DO 241 K=1,6
      WRITE(6,245) K,A,B,THET(K),H(K),MACH
  245 FORMAT(I5,5F20.7)
  241 CONTINUE
      SI=SMALA(0.5,4,1,4,2,1,3,1,4,1,5,7) * SIN(PHI(
     1 1,2,1,3,1,4,1,5,7,7)) + SMALA(0.125,4,1,4,2,2,4,1,5,1,7,7)
     2 * SIN(PHI(2,2,1,4,1,5,1,7,7,7)) + SMALA(0.25,4,1,4,2,2,
     3 3,1,5,1,7,7) * SIN(PHI(2,2,1,3,1,5,1,7,7,7))
      SII=SMALA(0.5,4,1,4,2,1,3,1,4,1,5,7) * COS(PHI(
     1 1,2,1,3,1,4,1,5,7,7)) + SMALA(0.125,4,1,4,2,2,4,1,5,1,7,7)
     2 * COS(PHI(2,2,1,4,1,5,1,7,7,7))+SMALA(0.25,4,1,4,2,2,
     3 3,1,5,1,7,7) * COS(PHI(2,2,1,3,1,5,1,7,7,7))
      WRITE(6,250)
  250 FORMAT(2H0,5X,2HIJ,10X,2HSI,18X,3HSII,17X,10HFACT(6,IJ),/)
      WRITE(6,255) IJ,SI,SII,FACT(6,IJ)
  255 FORMAT(I6,3F20.5)
      SCRIPE(IJ)=SQRT(SI**2 + SII**2)
      SCRIPE(IJ)=COS(THET(1))*SCRIPE(IJ)
      MAXA(IJ)=0.5*(AMPF**4)*(1./((DELT(1)**4)*DELT(2)*
     1 DELT(3)*DELT(4) *DELT(5)))
      QE(IJ)=SCRIPE(IJ)/MAXA(IJ)
      A=SI
      B=SII
      PHASE(5,IJ)=ATAN2(A,B)
   33 CONTINUE
      WRITE(6,135)
  135 FORMAT(1H1,4X,1HI,8X,5HWD(I),5X,9HSCRIPE(I),7X,7HMAXA(I),
     1 9X,5HQE(I),10X,10HPHASE(5,I),/)
      DO 39 I=1,41
      WRITE(6,145) I,WD(I),SCRIPE(I),MAXA(I),QE(I),PHASE(5,I)
  145 FORMAT(I6,5F20.8)
   39 CONTINUE
C
C
C     THIS PORTION OF THE PROGRAM COMPUTES THE AMPLITUDE AND PHASE OF
C     THE SIXTH HARMONIC.
C
C
      DO 41 KK=1,41
      WD(KK)=-5.0+(KK-1)*0.25
      DO 43 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(KK)*WR)-(WD(KK)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(KK)/WR))**2)
     1 + (DELT(K)**2))
      MACH = (2.*A0)/(X*PI*WR*C0*H(1))
   43 CONTINUE
      FACT(6,KK)=(((MACH*BETA)**6)*SQRT(720.))/(DELT(1)**6)
      TI=SMALA(0.5,5,1,5,2,1,3,1,4,1,5,6) * SIN(PHI(
     1 1,2,1,3,1,4,1,5,6,7)) + SMALA(0.125,5,1,5,2,2,4,1,5,1,
     2 6,7) * SIN(PHI(2,2,1,4,1,5,1,6,7,7)) + SMALA(0.25,5,
     3 1,5,2,2,3,1,5,1,6,7) * SIN(PHI(2,2,1,3,1,5,1,6,7,7))
     4 + SMALA(0.25,5,1,5,2,2,3,1,4,1,6,7) * SIN(PHI(
     5 2,2,1,3,1,4,1,6,7,7)) + SMALA(0.0625,5,1,5,2,3,4,1,6,1,
     6 7,7) * SIN(PHI(3,2,1,4,1,6,1,7,7,7)) + SMALA(0.125,5,
     7 1,5,2,2,3,2,6,1,7,7) * SIN(PHI(2,2,2,3,1,6,1,7,7,7))
      TII = SMALA(0.5,5,1,5,2,1,3,1,4,1,5,6) * COS(PHI(
     1 1,2,1,3,1,4,1,5,6,7)) + SMALA(0.125,5,1,5,2,2,4,1,5,1,
     2 6,7) * COS(PHI(2,2,1,4,1,5,1,6,7,7)) + SMALA(0.25,5,
     3 1,5,2,2,3,1,5,1,6,7) * COS(PHI(2,2,1,3,1,5,1,6,7,7))
     4 + SMALA(0.25,5,1,5,2,2,3,1,4,1,6,7) * COS(PHI(
     5 2,2,1,3,1,4,1,6,7,7)) + SMALA(0.0625,5,1,5,2,3,4,1,6,1,
     6 7,7) * COS(PHI(3,2,1,4,1,6,1,7,7,7)) + SMALA(0.125,5,
     7 1,5,2,2,3,2,6,1,7,7) * COS(PHI(2,2,2,3,1,6,1,7,7,7))
      SCRIPF(KK)=SQRT(TI**2+TII**2)
      SCRIPF(KK)=COS(THET(1))*SCRIPF(KK)
      MAXA(KK)=0.5*(AMPF**5)*(1./((DELT(1)**5)*DELT(2)*
     1 DELT(3) *DELT(4)*DELT(5)*DELT(6)))
      DF(KK)=SCRIPF(KK)/MAXA(KK)
      A=TI
      B=TII
      PHASE(6,KK)=ATAN2(A,B)
   41 CONTINUE
```

```
      WRITE(6,155)
  155 FORMAT(1H1,4X,1H1,8X,5HWD(I),5X,9HSCRIPF(I),7X,7HMAXA(I),
     1 9X,5HQF(L),10X,10HPHASE(6,L),/)
      DO 47 M=1,41
      WRITE(6,165) M,WD(M),SCRIPF(M),MAXA(M),QF(M),PHASE(6,M)
  165 FORMAT(I6,5F20.8)
   47 CONTINUE
      STOP
      END
      FUNCTION SMALA(D,J,K,L,M,II,JJ,KK,LL,MM,IJ,IK)
      DIMENSION THET(7),H(7)
      COMMON THET,H,AMPF
      H(7)=1.
      SMALA=D*AMPF**J*(1./((H(K)**L)*(H(M)**II) *
     1 (H(JJ)**KK) * (H(LL)**MM) * H(IJ) * H(IK)))
      RETURN
      END
```

```
      FUNCTION PHI(I,J,K,L,M,IJ,IK,IL,IM,II)
      DIMENSION THET(7),H(7)
      COMMON THET,H,AMPF
      THET(7)=0.0
      PHI=I*THET(J)+K*THET(L)+M * THET(IJ)
     1 + IK * THET(IL)+THET(IM)+THET(II)
      WRITE(6,700) PHI
  700 FORMAT(F30.15)
      RETURN
      END
```

B-6: Program PHAMPC

This program is identical to PHAMP except that it was written for the Fourier synthesis approach and graphs the phase angles associated with each harmonic. In order to use the graphing subroutine the subscripts on the phase angle had to be reversed. PHASE(I,J) is now defined as the phase angle associated with the i-th $\Delta\omega$ and the j-th harmonic. In all other respects Table B.2 is applicable.

```
C
C
C     THIS PROGRAM IS DESIGNED TO COMPUTE THE AMPLITUDE AND PHASE
C     OF EACH HARMONIC AS THE FREQUENCY IS SWEPT THROUGH RESONANCE.
C     EACH AMPLITUDE IS PRINTED OUT IN BOTH A NORMALIZED AND AN
C     UN-NORMALIZED FORM. THE NORMALIZED FORM HAS BEEN NORMALIZED BY
C     THE AMPLITUDE OF THE FIRST TERM OF THAT HARMONIC.
C
C
      DIMENSION DELT(6),THET(7),H(7),SCRIPA(81),SCRIPB(81),SCRIPC(81),
     1 SCRIPD(81),SCRIPE(81),SCRIPF(81),MAXA(81),WD(81),QA(81),QB(81),
     2 QC(81),QD(81),QE(81),QF(81),FACT(6,81),PHASE(81,6)
      COMMON THET,H,AMPF
      REAL MACH
      REAL MAXA
      REAL LABEL/4H    /
      REAL*8 ITITLE(12)/'PHAMPC  ','P. G. RU','FF BOX R',
     1 'PHASE AN','GLE VS D','ELTA W  ','    AO= ','        ',
     2 '        ','HARMONIC','        ','        '/
      READ(5,10) DELT(1),WR,BETA,X,AO,F
   10 FORMAT(6F10.5)
      WRITE(6,15)
   15 FORMAT(1H1,2X,7HDELT(1),5X,2HWR,7X,4HBETA,7X,1HX,8X,2HAO,9X,
     1 1HF,/)
      WRITE(6,25) DELT(1),WR,BETA,X,AO,F
   25 FORMAT(6F10.5)
      PI=3.1416
      CO = 343.4
      C=PI/F
      AMPF=(AO*BETA)/(X*PI*WR*CO)
      DO 17 I=1,6
      TEMP=I
      DELT(I)=DELT(1)/SQRT(TEMP)
   17 CONTINUE
      WRITE(6,175)
  175 FORMAT(1H0,10X,1HI,10X,1HC,10X,4HAMPF,10X,7HDELT(I),/)
      DO 57 I=1,6
      WRITE(6,185) I,C,AMPF,DELT(I)
  185 FORMAT(I11,F16.5,F13.10,F15.7)
   57 CONTINUE
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
```

B-6: Program PHAMPC

This program is identical to PHAMP except that it was written for the Fourier synthesis approach and graphs the phase angles associated with each harmonic. In order to use the graphing subroutine the subscripts on the phase angle had to be reversed. PHASE(I,J) is now defined as the phase angle associated with the i-th $\Delta\omega$ and the j-th harmonic. In all other respects Table B.2 is applicable.

```
C
C
C     THIS PROGRAM IS DESIGNED TO COMPUTE THE AMPLITUDE AND PHASE
C     OF EACH HARMONIC AS THE FREQUENCY IS SWEPT THROUGH RESONANCE.
C     EACH AMPLITUDE IS PRINTED OUT IN BOTH A NORMALIZED AND AN
C     UN-NORMALIZED FORM. THE NORMALIZED FORM HAS BEEN NORMALIZED BY
C     THE AMPLITUDE OF THE FIRST TERM OF THAT HARMONIC.
C
C
      DIMENSION DELT(6),THET(7),H(7),SCRIPA(81),SCRIPB(81),SCRIPC(81),
     1 SCRIPD(81),SCRIPE(81),SCRIPF(81),MAXA(81),WD(81),QA(81),QB(81),
     2 QC(81),QD(81),QE(81),QF(81),FACT(6,81),PHASE(81,6)
      COMMON THET,H,AMPF
      REAL MACH
      REAL MAXA
      REAL LABEL/4H    /
      REAL*8 ITITLE(12)/'PHAMPC  ','P. G. RU','FF BOX R',
     1 'PHASE AN','GLE VS D','ELTA W  ','    AO= ','        ',
     2 '        ','HARMONIC','        ','        '/
      READ(5,10) DELT(1),WR,BETA,X,AO,F
   10 FORMAT(6F10.5)
      WRITE(6,15)
   15 FORMAT(1H1,2X,7HDELT(1),5X,2HWR,7X,4HBETA,7X,1HX,8X,2HAO,9X,
     1 1HF,/)
      WRITE(6,25) DELT(1),WR,BETA,X,AO,F
   25 FORMAT(6F10.5)
      PI=3.1416
      CO = 343.4
      C=PI/F
      AMPF=(AO*BETA)/(X*PI*WR*CO)
      DO 17 I=1,6
      TEMP=I
      DELT(I)=DELT(1)/SQRT(TEMP)
   17 CONTINUE
      WRITE(6,175)
  175 FORMAT(1HO,10X,1HI,10X,1HC,10X,4HAMPF,10X,7HDELT(I),/)
      DO 57 I=1,6
      WRITE(6,185) I,C,AMPF,DELT(I)
  185 FORMAT(I11,F16.5,F13.10,F15.7)
   57 CONTINUE
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
```

```
C      FOR THE FIRST HARMONIC
C
C
       DO 9 M=1,81
       WD(M)=-5.0+(M-1)*0.125
       DO 3 K=1,6
       A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(M)*WR)-(WD(M)**2)
       B=DELT(K)*(WR**2)
       THET(K)=ATAN2(A,B)
       H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(M)/WR))**2)+(DELT(K)**2))
       MACH=(2.*AO)/(X*PI*WR*CO*H(1))
     3 CONTINUE
       FACT(1,M)=(((MACH*BETA)**1)*SQRT(1.))/(DELT(1)**1)
       WRITE(6,230)
   230 FORMAT(1H0,4X,1HK,10X,1HA,18X,1HB,16X,7HTHET(K),15X,4HH(K),
      1 18X,4HMACH,/)
       DO 211 K=1,6
       WRITE(6,220) K,A,B,THET(K),H(K),MACH
   220 FORMAT(I5,5F20.7)
   211 CONTINUE
       SCRIPA(M) = COS(THET(1))
       MAXA(M)=1.
       QA(M)=SCRIPA(M)/MAXA(M)
       A=0.0
       B=1.0
       PHASE(M,1)=ATAN2(A,B)
     9 CONTINUE
       WRITE(6,55)
    55 FORMAT(1H1,4X,1HM,8X,5HWD(M),5X,9HSCRIPA(M),7X,7HMAXA(M),9X,
      1 5HQA(M),10X,10HPHASE(M,1),/)
       DO 11 M=1,81
       WRITE(6,65) M,WD(M),SCRIPA(M),MAXA(M),QA(M),PHASE(M,1)
    65 FORMAT(I6,5F20.8)
    11 CONTINUE
       READ(5,740) ITITLE(8),ITITLE(9)
   740 FORMAT(2A8)
       CALL DRAW(81,WD,PHASE(1,1),0,0,LABEL,ITITLE,0,0,0,0,0,0,6,8,0,L)
C
C
C      THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C      FOR THE SECOND HARMONIC.
C
C
```

```
      DO 13 J=1,81
      WD(J)=-5.0+(J-1)*0.125
      DO 59 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(J)*WR)-(WD(J)**2)
      B=DELT(K) * (WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(J)/WR))**2)
     1 +(DELT(K)**2))
      MACH =  (2.*AO)/(X*PI*WR*CO*H(1))
   59 CONTINUE
      FACT(2,J)=(((MACH*BETA)**2)*SQRT(2.))/(DELT(1)**2)
      GI=SMALA(0.5,1,1,1,2,1,7,1,7,1,7,7)  * SIN(PHI
     1 (1,2,1,7,1,7,1,7,7,7))
      GD=SMALA(0.5,1,1,1,2,1,7,1,7,1,7,7)  * COS(PHI(
     1 1,2,1,7,1,7,1,7,7,7))
      GII = GD
      SCRIPB(J)=(COS(THET(1)))*(SQRT((GI**2)+(GII**2)))
      MAXA(J)=(0.5 * AMPF)/(DELT(1)*DELT(2))
      QB(J)=SCRIPB(J)/MAXA(J)
      A=GI
      B=GII
      PHASE(J,2)=ATAN2(A,B)
   13 CONTINUE
      WRITE(6,75)
   75 FORMAT(1H1,4X,1HJ,8X,5HWD(J),5X,9HSCRIPB(J),7X,7HMAXA(J),9X,
     1 5HQB(J),10X,10HPHASE(J,2),/)
      DO 19 J=1,81
      WRITE(6,85) J,WD(J),SCRIPB(J),MAXA(J),QB(J),PHASE(J,2)
   85 FORMAT(I6,5F20.8)
   19 CONTINUE
      READ(5,750) ITITLE(8),ITITLE(9)
  750 FORMAT(2A8)
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE THIRD HARMONIC.
C
C
      DO 21 L=1,81
      WD(L)=-5.0+(L-1)*0.125
      DO 23 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(L)*WR)-(WD(L)**2)
      B=DELT(K) * (WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(L)/WR))**2)
     1 +(DELT(K)**2))
      MACH = (2.*AO)/(X*PI*WR*CO*H(1))
   23 CONTINUE
      FACT(3,L) = (((MACH*BETA)**3)*SQRT(6.))/(DELT(1)**3)
      HI=SMALA(0.5,2,1,2,2,1,3,1,7,1,7,7) * SIN(PHI(
     1 1,2,1,3,1,7,1,7,7,7))
      HII=SMALA(0.5,2,1,2,2,1,3,1,7,1,7,7) * COS(PHI(
     1 1,2,1,3,1,7,1,7,7,7))
      SCRIPC(L)=(COS(THET(1)))*(SQRT((HI**2)+(HII**2)))
      MAXA(L)=0.5*(AMPF**2)*(1./((DELT(1)**2)*DELT(2)*DELT(3)))
      QC(L)=SCRIPC(L)/MAXA(L)
      A=HI
      B=HII
      PHASE(L,3)=ATAN2(A,B)
   21 CONTINUE
      WRITE(6,95)
   95 FORMAT(1H1,4X,1HL,8X,5HWD(L),5X,9HSCRIPC(L),7X,7HMAXA(L),
     1 9X,5HQC(L),10X,10HPHASE(L,3),/)
      DO 67 II=1,81
      WRITE(6,105) II,WD(II),SCRIPC(II),MAXA(II),QC(II),PHASE(II,3)
  105 FORMAT(I6,5F20.8)
   67 CONTINUE
      READ(5,760) ITITLE(8),ITITLE(9)
  760 FORMAT(2A8)
      CALL DRAW(81,WD,PHASE(1,3),0,0,LABEL,ITITLE,0,0,0,0,0,0,6,8,0,L)
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE FOURTH HARMONIC.
C
C
      DO 27 JJ=1,81
      WD(JJ)=-5.0+(JJ-1)*0.125
      DO 29 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(JJ)*WR)-(WD(JJ)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(JJ)/WR))**2)
     1 + (DELT(K)**2))
      MACH = (2.*AO)/(X*PI*WR*CO*H(1))
   29 CONTINUE
```

```
      WRITE(6,240)
  240 FORMAT(1H0,4X,1HK,10X,1HA,18X,1HB,16X,7HTHET(K),15X,4HH(K),
     1 18X,4HMACH,/)
      DO 241 K=1,6
      WRITE(6,245) K,A,B,THET(K),H(K),MACH
  245 FORMAT(I5,5F20.7)
  241 CONTINUE
      RI=SMALA(0.5,3,1,3,2,1,3,1,4,1,7,7)* SIN(PHI(
     1 1,2,1,3,1,4,1,7,7,7)) + SMALA(0.125,3,1,3,2,2,4,1,7,1,7,7)
     2 * SIN(PHI(2,2,1,4,1,7,1,7,7,7))
      RC=SMALA(0.5,3,1,3,2,1,3,1,4,1,7,7) * COS(PHI(
     1 1,2,1,3,1,4,1,7,7,7)) + SMALA(0.125,3,1,3,2,2,4,1,7,1,7,7)
     2 * COS(PHI(2,2,1,4,1,7,1,7,7,7))
      RII = RC
      SCRIPD(JJ)=(COS(THET(1)))*(SQRT((RI**2)+(RII**2)))
      MAXA(JJ)=0.5*(AMPF**3)*(1./((DELT(1)**3)*DELT(2)*DELT(3)*
     1 DELT(4)))
      QD(JJ)=SCRIPD(JJ)/MAXA(JJ)
      A=RI
      B=RII
      PHASE(JJ,4)=ATAN2(A,B)
   27 CONTINUE
      WRITE(6,115)
  115 FORMAT(1H1,4X,1HK,8X,5HWD(K),5X,9HSCRIPD(K),7X,7HMAXA(K),
     1 9X,5HQD(K),10X,10HPHASE(K,4),/)
      DO 31 K=1,81
      WRITE(6,125) K,WD(K),SCRIPD(K),MAXA(K),QD(K),PHASE(K,4)
  125 FORMAT(I6,5F20.8)
   31 CONTINUE
      READ(5,770) ITITLE(8),ITITLE(9)
  770 FORMAT(2A8)
      CALL DRAW(81,WD,PHASE(1,4),0,0,LABEL,ITITLE,0,0,0,0,0,0,6,8,0,L)
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE FIFTH HARMONIC
C
C
      DO 33 IJ=1,81
      WD(IJ)=-5.0+(IJ-1)*0.125
      DO 37 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(IJ)*WR)-(WD(IJ)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(IJ)/WR))**2)+(DELT(K)**2))
      MACH = (2.*AO)/(X*PI*WR*CO*H(1))
   37 CONTINUE
      SI=SMALA(0.5,4,1,4,2,1,3,1,4,1,5,7) * SIN(PHI(
     1 1,2,1,3,1,4,1,5,7,7)) + SMALA(0.125,4,1,4,2,2,4,1,5,1,7,7)
     2 * SIN(PHI(2,2,1,4,1,5,1,7,7,7)) + SMALA(0.25,4,1,4,2,2,
     3 3,1,5,1,7,7) * SIN(PHI(2,2,1,3,1,5,1,7,7,7))
      SII=SMALA(0.5,4,1,4,2,1,3,1,4,1,5,7) * COS(PHI(
     1 1,2,1,3,1,4,1,5,7,7)) + SMALA(0.125,4,1,4,2,2,4,1,5,1,7,7)
     2 * COS(PHI(2,2,1,4,1,5,1,7,7,7))+SMALA(0.25,4,1,4,2,2,
     3 3,1,5,1,7,7) * COS(PHI(2,2,1,3,1,5,1,7,7,7))
      SCRIPE(IJ)=(COS(THET(1)))*(SQRT((SI**2)+(SII**2)))
      MAXA(IJ)=0.5*(AMPF**4)*(1./((DELT(1)**4)*DELT(2)*
     1 DELT(3)*DELT(4) *DELT(5)))
      QE(IJ)=SCRIPE(IJ)/MAXA(IJ)
      A=SI
      B=SII
      PHASE(IJ,5)=ATAN2(A,B)
   33 CONTINUE
      WRITE(6,135)
  135 FORMAT(1H1,4X,1HI,8X,5HWD(I),5X,9HSCRIPE(I),7X,7HMAXA(I),
     1 9X,5HQE(I),10X,10HPHASE(I,5),/)
      DO 39 I=1,81
      WRITE(6,145) I,WD(I),SCRIPE(I),MAXA(I),QE(I),PHASE(I,5)
  145 FORMAT(I6,5F20.8)
   39 CONTINUE
      READ(5,780) ITITLE(8),ITITLE(9)
  780 FORMAT(2A8)
      CALL DRAW(81,WD,PHASE(1,5),0,0,LABEL,ITITLE,0,0,0,0,0,0,6,8,0,L)
C
C
C     THIS PORTION COMPUTES AND GRAPHS THE NORMALIZED Q CURVE
C     FOR THE SIXTH HARMONIC.
C
C
      DO 41 KK = 1,81
      WD(KK)=-5.0+(KK-1)*0.125
      DO 43 K=1,6
      A=(DELT(1)-DELT(K))*(WR**2)-(2.*WD(KK)*WR)-(WD(KK)**2)
      B=DELT(K)*(WR**2)
      THET(K)=ATAN2(A,B)
      H(K)=SQRT(((DELT(1)-DELT(K)-(2.*WD(KK)/WR))**2)
     1 + (DELT(K)**2))
      MACH = (2.*AO)/(X*PI*WR*CO*H(1))
   43 CONTINUE
      FACT(6,KK)=(((MACH*BETA)**6)*SQRT(720.))/(DELT(1)**6)
      TI=SMALA(0.5,5,1,5,2,1,3,1,4,1,5,6) * SIN(PHI(
     1 1,2,1,3,1,4,1,5,6,7)) + SMALA(0.125,5,1,5,2,2,4,1,5,1,
     2 6,7) * SIN(PHI(2,2,1,4,1,5,1,6,7,7)) + SMALA(0.25,5,
     3 1,5,2,2,3,1,5,1,6,7) * SIN(PHI(2,2,1,3,1,5,1,6,7,7))
     4 + SMALA(0.25,5,1,5,2,2,3,1,4,1,6,7) * SIN(PHI(
     5 2,2,1,3,1,4,1,6,7,7)) + SMALA(0.0625,5,1,5,2,3,4,1,6,1,
     6 7,7) * SIN(PHI(3,2,1,4,1,6,1,7,7,7)) + SMALA(0.125,5,
     7 1,5,2,2,3,2,6,1,7,7) * SIN(PHI(2,2,2,3,1,6,1,7,7,7))
      TII = SMALA(0.5,5,1,5,2,1,3,1,4,1,5,6) * COS(PHI(
     1 1,2,1,3,1,4,1,5,6,7)) + SMALA(0.125,5,1,5,2,2,4,1,5,1,
     2 6,7) * COS(PHI(2,2,1,4,1,5,1,6,7,7)) + SMALA(0.25,5,
     3 1,5,2,2,3,1,5,1,6,7) * COS(PHI(2,2,1,3,1,5,1,6,7,7))
     4 + SMALA(0.25,5,1,5,2,2,3,1,4,1,6,7) * COS(PHI(
     5 2,2,1,3,1,4,1,6,7,7)) + SMALA(0.0625,5,1,5,2,3,4,1,6,1,
     6 7,7) * COS(PHI(3,2,1,4,1,6,1,7,7,7)) + SMALA(0.125,5,
     7 1,5,2,2,3,2,6,1,7,7) * COS(PHI(2,2,2,3,1,6,1,7,7,7))
      SCRIPF(KK)=(COS(THET(1)))*(SQRT((TI**2)+(TII**2)))
      MAXA(KK)=0.5*(AMPF**5)*(1./((DELT(1)**5)*DELT(2)*
     1 DELT(3) *DELT(4)*DELT(5)*DELT(6)))
      QF(KK)=SCRIPF(KK)/MAXA(KK)
      A=TI
      B=TII
      PHASE(KK,6)=ATAN2(A,B)
   41 CONTINUE
      WRITE(6,155)
  155 FORMAT(1H1,4X,1HI,8X,5HWD(I),5X,9HSCRIPF(I),7X,7HMAXA(I),
     1 9X,5HQF(I),10X,10HPHASE(I,6),/)
      DO 47 M=1,81
      WRITE(6,165) M,WD(M),SCRIPF(M),MAXA(M),QF(M),PHASE(M,6)
  165 FORMAT(I6,5F20.8)
   47 CONTINUE
      READ(5,790) ITITLE(8),ITITLE(9)
  790 FORMAT(2A8)
      CALL DRAW(81,WD,PHASE(1,6),0,0,LABEL,ITITLE,0,0,0,0,0,0,6,8,0,L)
      STOP
      END
      FUNCTION SMALA(D,J,K,L,M,II,JJ,KK,LL,MM,IJ,IK)
      COMMON THET,H,AMPF
      DIMENSION THET(7),H(7)
      H(7)=1.
      SMALA=D*AMPF**J*(1./((H(K)**L)*(H(M)**II) *
     1 (H(JJ)**KK) * (H(LL)**MM) * H(IJ) * H(IK)))
      RETURN
      END
```

```
      FUNCTION PHI(I,J,K,L,M,IJ,IK,IL,IM,II)
      DIMENSION THET(7),H(7)
      COMMON THET,H,AMPF
      THET(7)=0.0
      PHI=I*THET(J)+K*THET(L)+M * THET(IJ)
     1 + IK * THET(IL)+THET(IM)+THET(II)
      WRITE(6,600) PHI
  600 FORMAT(F30.15)
      RETURN
      END
```

B-7: Program FOUANAL

This program is an adaptation of work done by Roy M. Johnson, Assistant Professor, Naval Postgraduate School. Basically the program takes a waveform obtained directly from the cavity and performs a Fourier analysis on it. The output of interest is a set of Fourier coefficients and their associated phase angles. The program is written in such a manner that a table of definitions is not appropriate. Each variable is defined within the program.

```
      COMPLEX*8A(1024)
      DIMENSION B(1024),ARG(1024),S(200),INV(200),M(3)
      TIME=ITIME(0)*0.01
      CALL CANCEL(2)
      WRITE(6,10)
      PI=3.1415927
      M(1)=6
      M(2)=0
      M(3)=0
      N1=2**M(1)
      DO 2 I1=1,N1
    2 A(I1)=(0.0,0.0)
      READ(5,100)(A(I1),I1=1,64)
  100 FORMAT(8F10.5)
      WRITE(6,40)(I1,A(I1),B(I1),ARG(I1),I1=1,N1)
      CLOCK=ITIME(0)*0.01
      CALL HARM(A,M,INV,S,-1,IFERR)
      CLOCK=ITIME(0)*0.01-CLOCK
      WRITE(6,30) IFERR
      DO 4 I1=1,N1
      IF(CABS(A(I1)).LT.1.0E-06)A(I1)=(0.0,0.0)
      B(I1)=CABS(A(I1))
      AR=REAL(A(I1))
      AI=AIMAG(A(I1))
      IF(AR.EQ.0.)AR=1.0E-10
      ARG(I1)=57.29578*ATAN2(AI,AR)
    4 CONTINUE
      WRITE(6,50)(I1,A(I1),B(I1),ARG(I1),I1=1,N1)
      WRITE(6,9999)CLOCK
      DO 6 I1=1,N1
      FI1=I1-1
      FI1=FI1/16.
      A(I1)=A(I1)*(0.0,1.0)*2.0*PI*FI1
      B(I1)=CABS(A(I1))
      AR=REAL(A(I1))
      AI=AIMAG(A(I1))
      IF(AR.EQ.0.)AR=1.0E-10
      ARG(I1)=57.29578*ATAN2(AI,AR)
    6 CONTINUE
      WRITE(6,40)(I1,A(I1),B(I1),ARG(I1),I1=1,N1)
      CALL HARM(A,M,INV,S,1,IFERR)
      DO 7 I1=1,N1
      B(I1)=CABS(A(I1))
      AR=REAL(A(I1))
      AI=AIMAG(A(I1))
      IF(AR.EQ.0.)AR=1.0E-10
      ARG(I1)=57.29578*ATAN2(AI,AR)
    7 CONTINUE
      WRITE(6,50)(I1,A(I1),B(I1),ARG(I1),I1=1,N1)
      WRITE(6,30) IFERR
      TIME=ITIME(0)*0.01-TIME
      WRITE(6,9999)TIME
   10 FORMAT(1H1)
   30 FORMAT(' IFERR=',I4)
   40 FORMAT(/' DATA COEFFICIENTS'//(2(1X,'A(',I4,')=(',1P1F10.3,',','1P1E
     &10.3,')=',1P1F10.3,' @ ',0P1F8.2,' DEG. ')))
   50 FORMAT(/' FOURIER COEFFICIENTS'//(2(1X,'X(',I4,')=(',1P1F10.3,',','1
     &P1E10.3,')=',1P1E10.3,' @ ',0P1F8.2,' DEG. ')))
 9999 FORMAT(/' ELAPSED COMPUTER TIME=',F7.2,' SEC.')
      END
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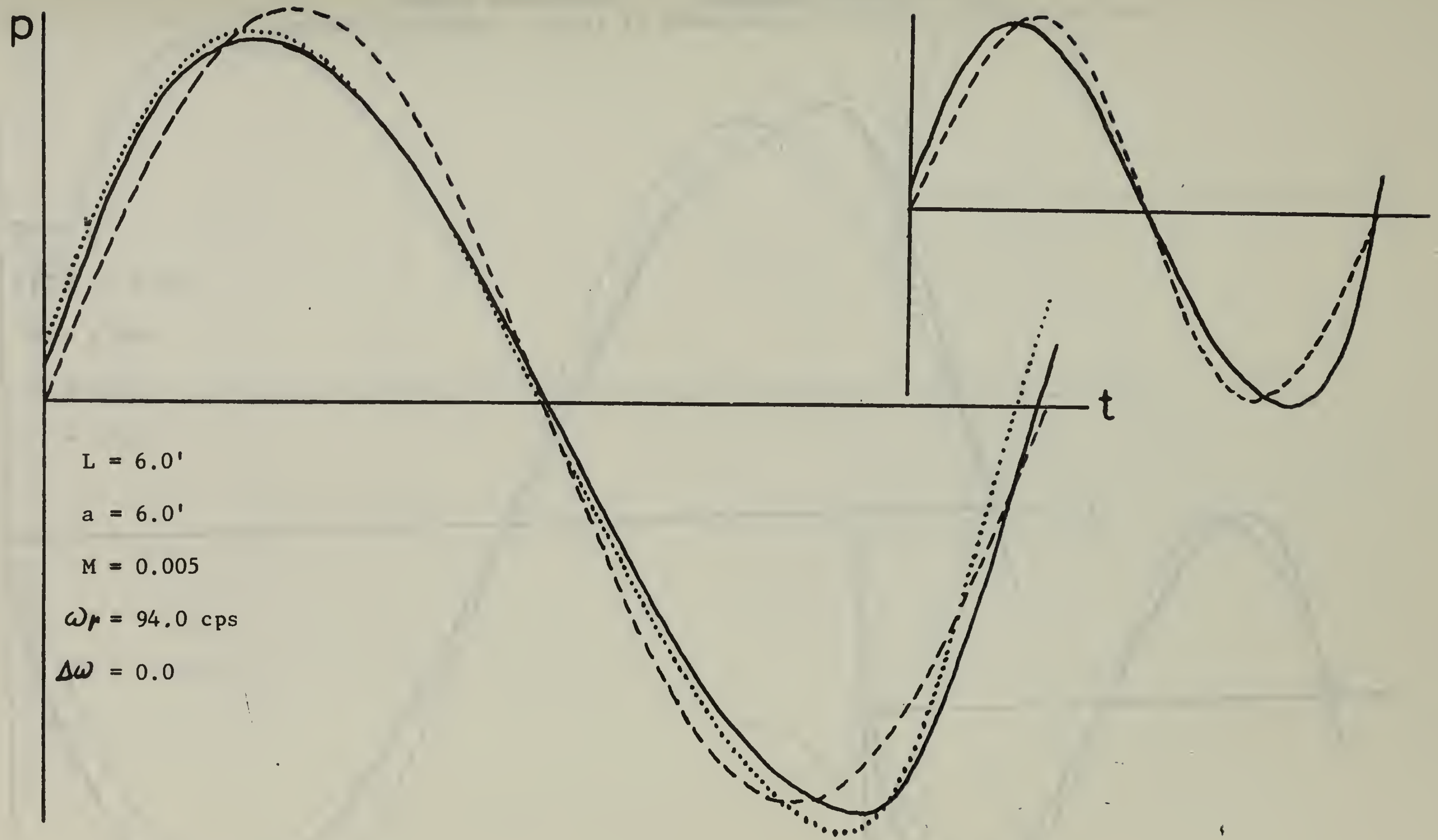
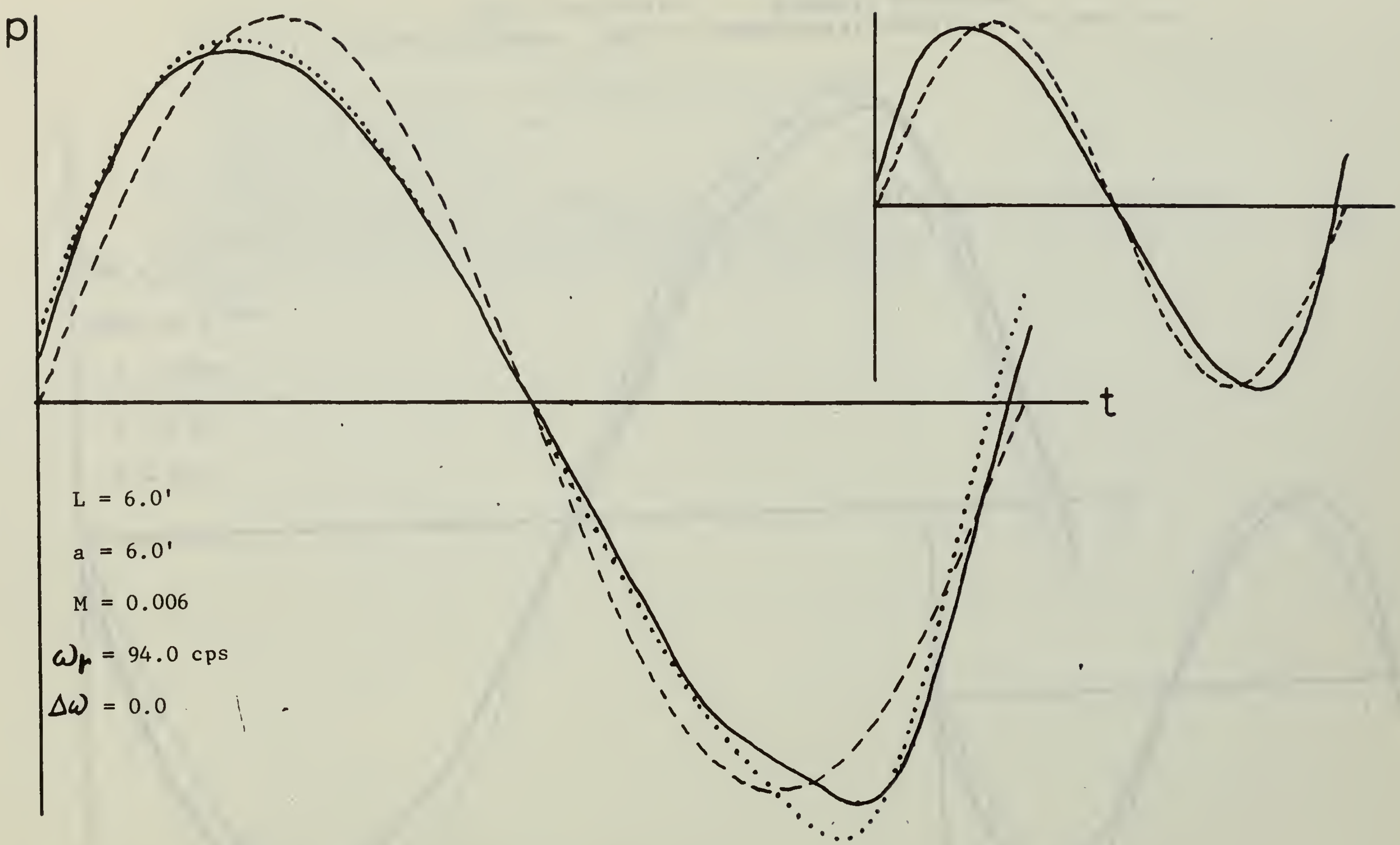


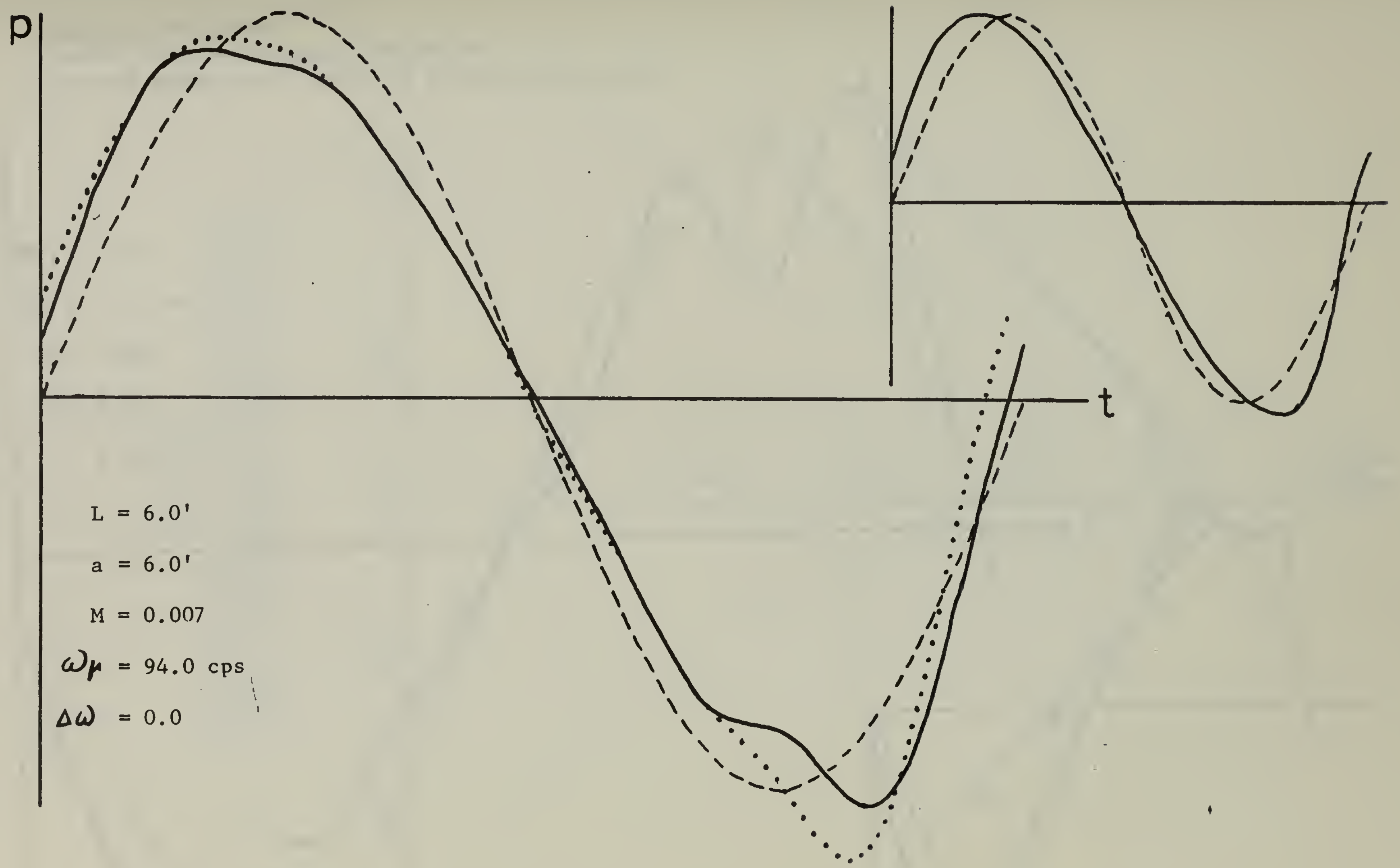
Pressure Waveforms Inset is experimental result,---sine wave,— FINAMPI prediction,..... FINAMPIZ prediction

Figure C.1



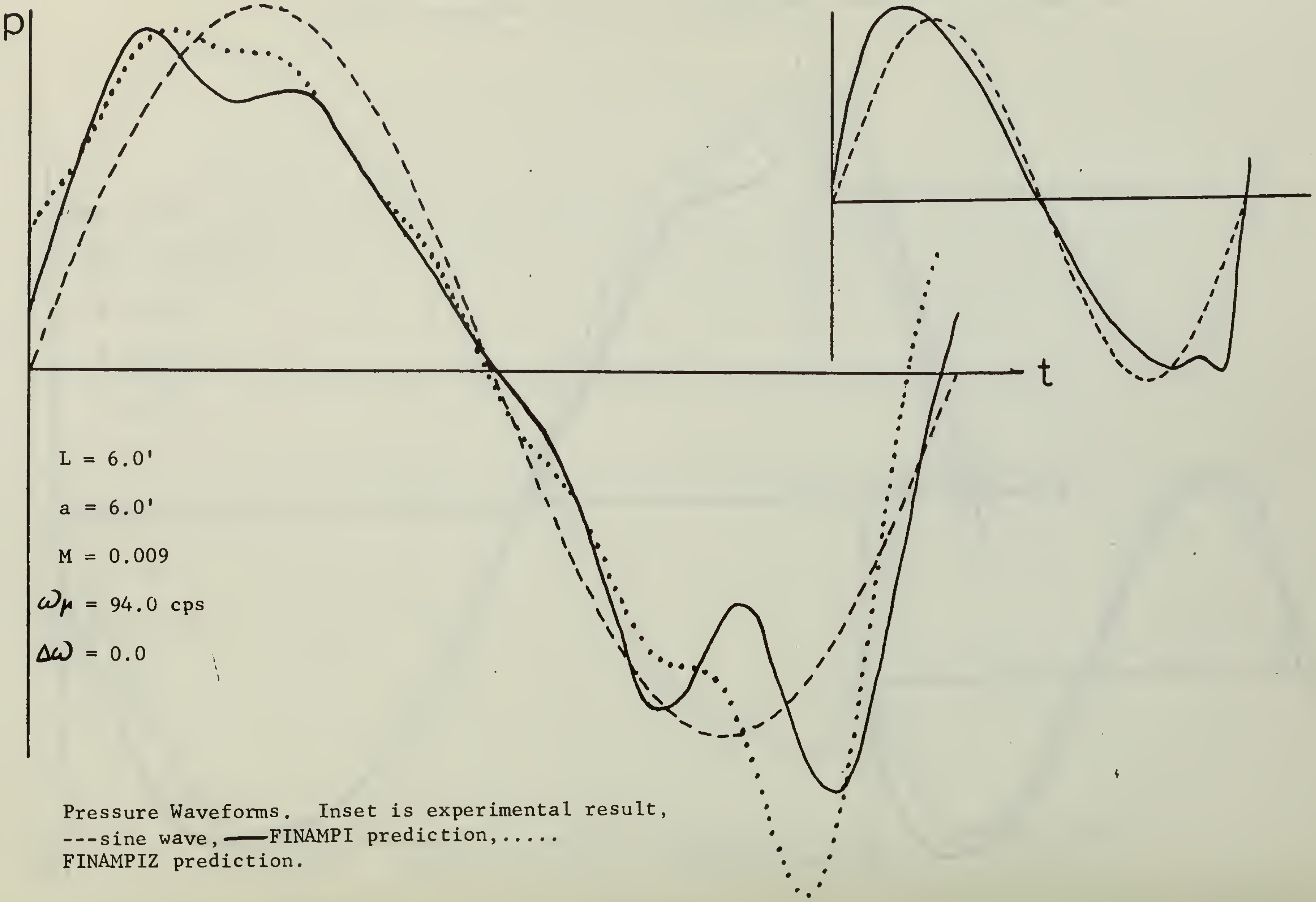
Pressure Waveforms Inset is experimental result,--- sine wave, —
FINAMPI prediction,.....FINAMPIZ prediction

Figure C.2



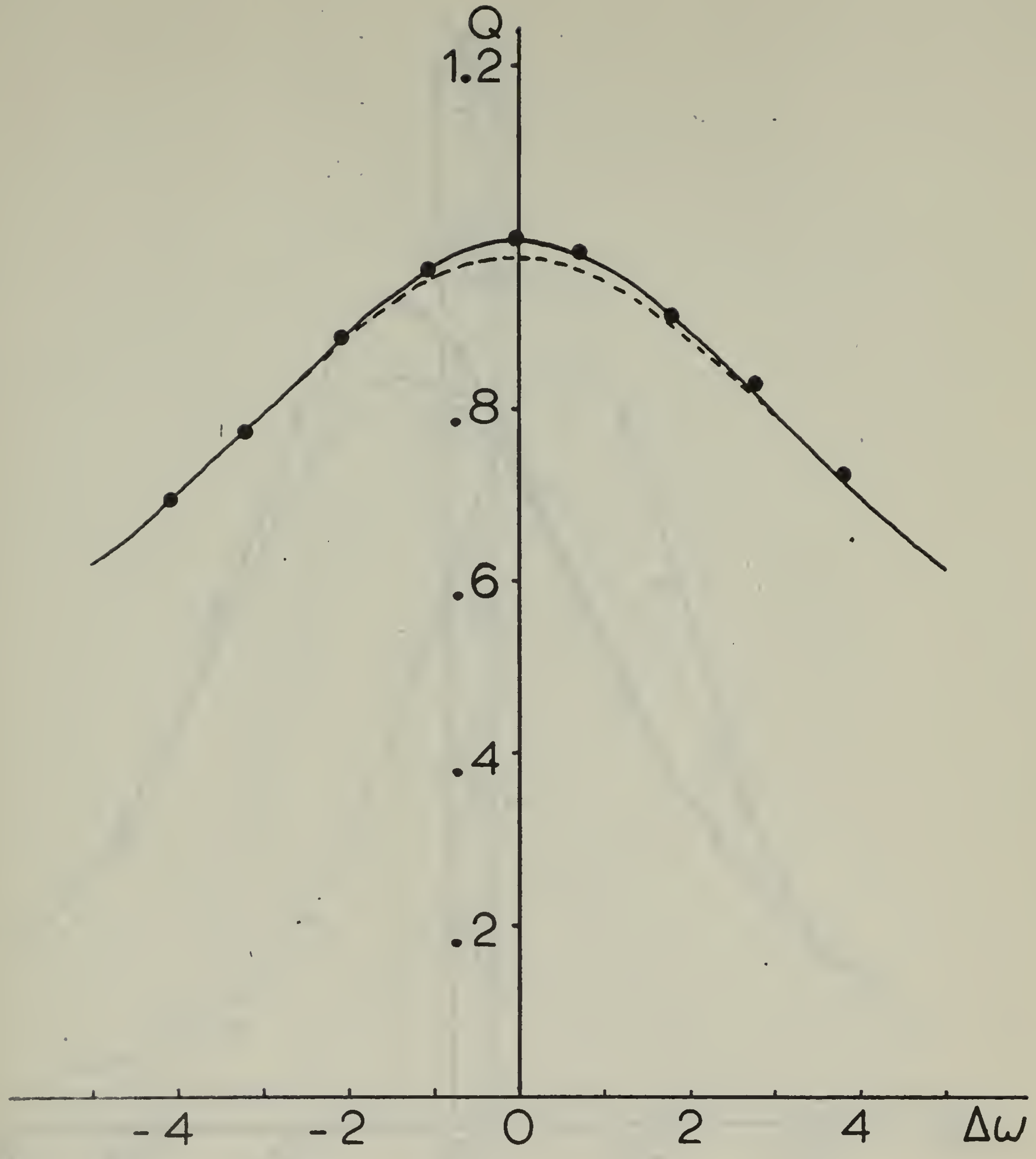
Pressure waveforms Inset is experimental result,--- sine wave, —— FINAMPI prediction,.....FINAMPIZ prediction

Figure C.3



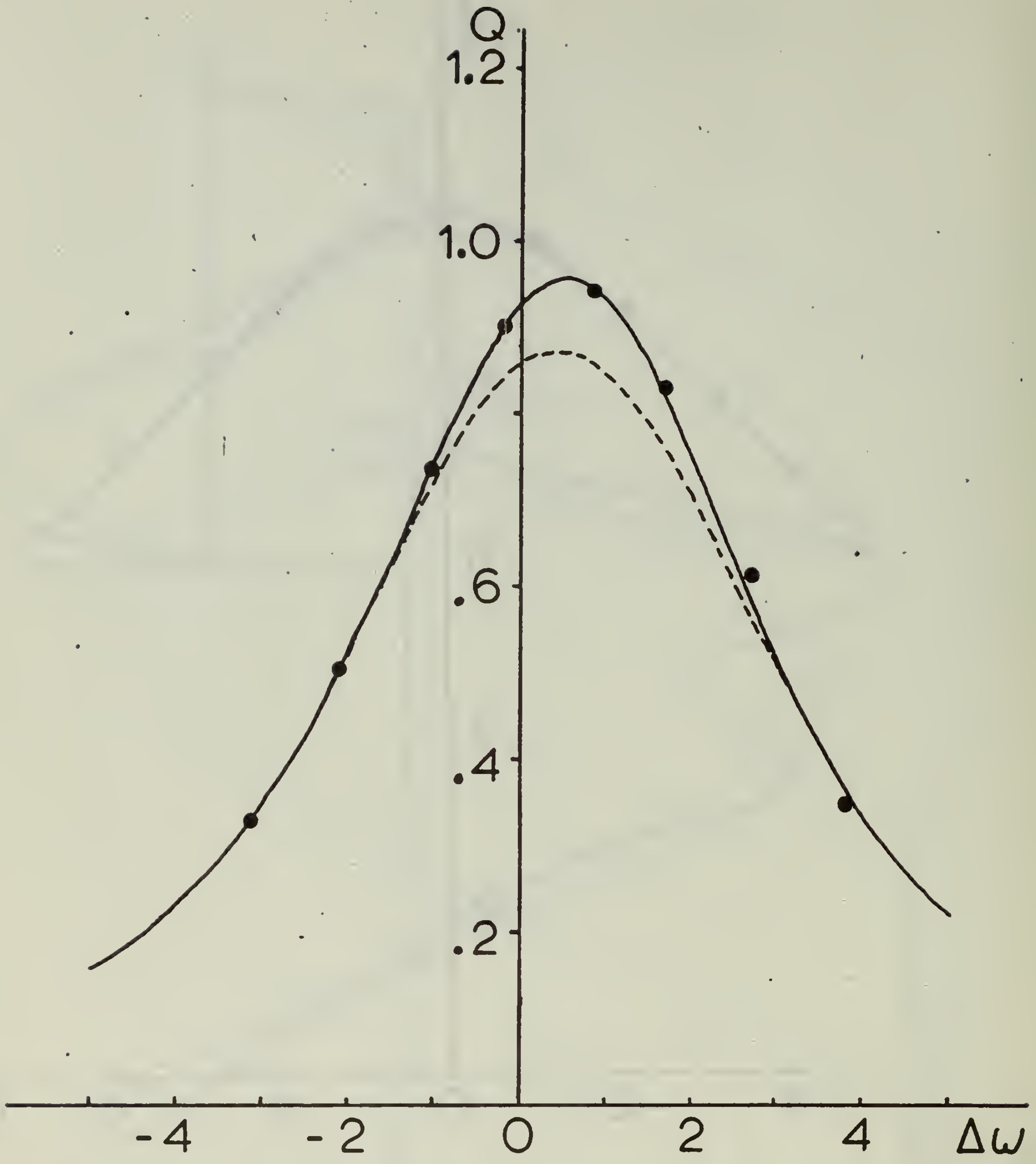
Pressure Waveforms. Inset is experimental result, ---sine wave, ——FINAMPI prediction, ... FINAMPIZ prediction.

Figure C.4



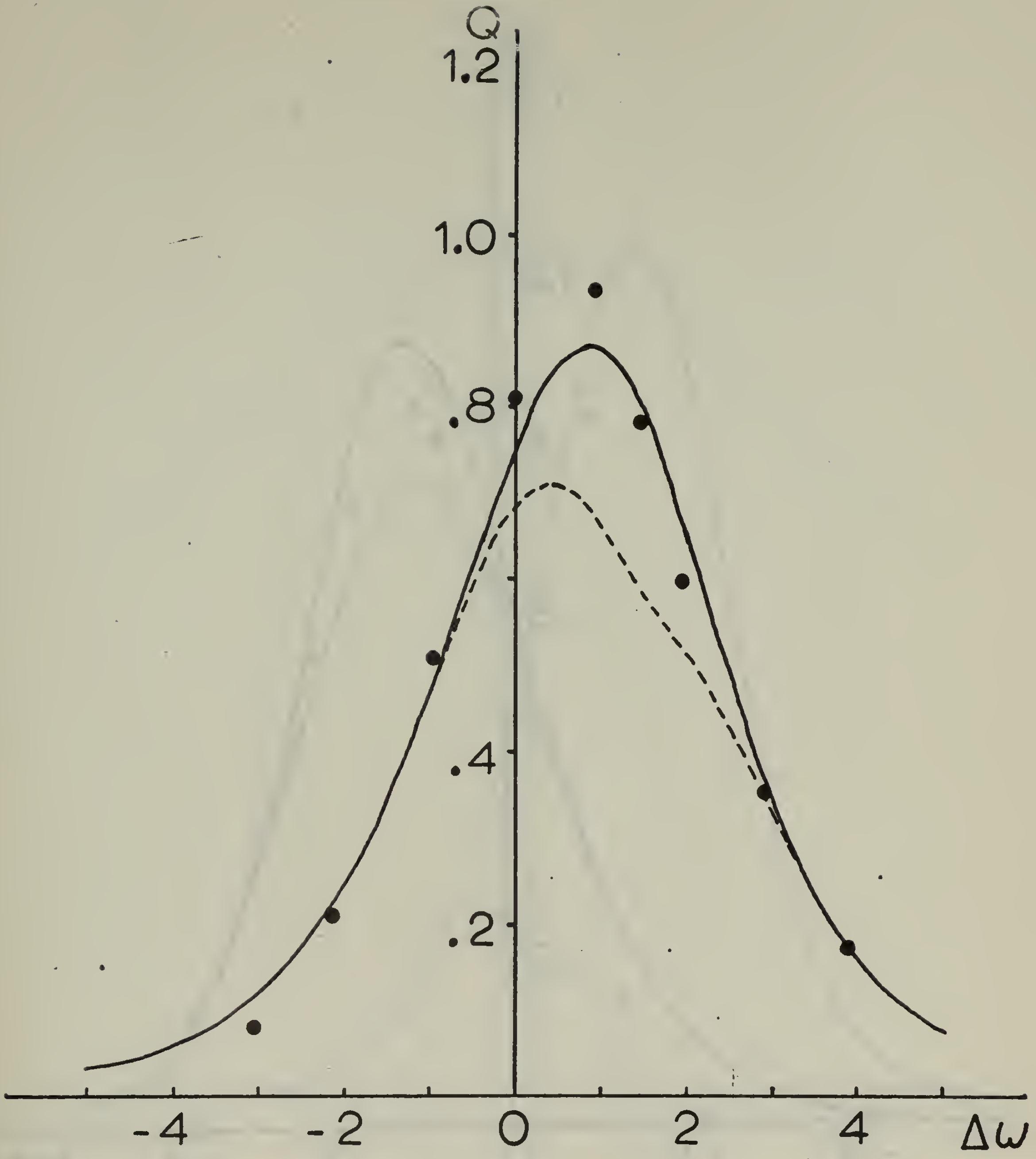
Q-curves for the fundamental, M = 0.004,
———QCURC prediction,---QCURVES prediction
Experimental results are indicated by ●.

Figure C.5



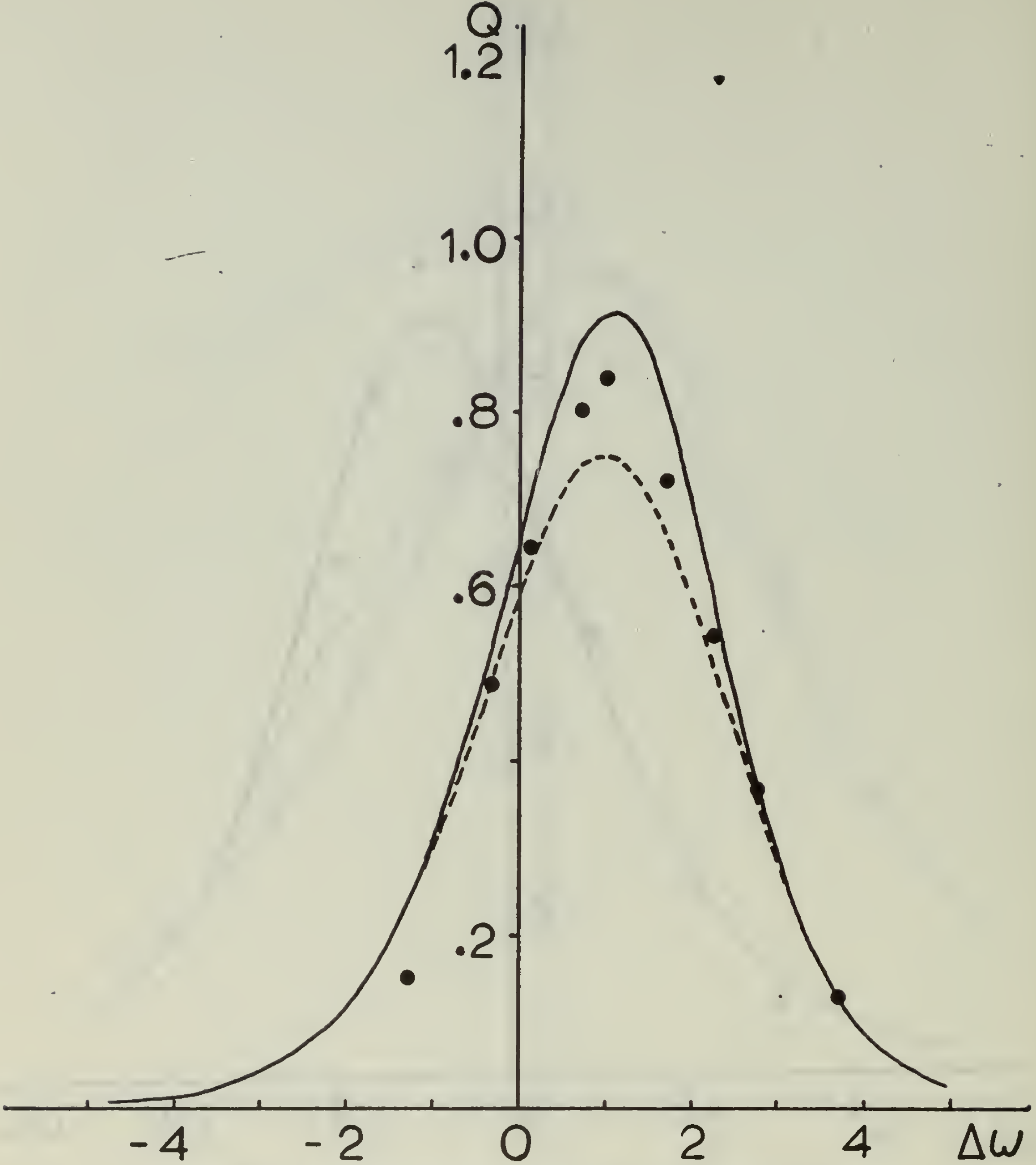
Q-curves for the second harmonic, M = 0.004
———QCURC prediction,----QCURVES prediction
Experimental results are indicated by ●.

Figure C.6



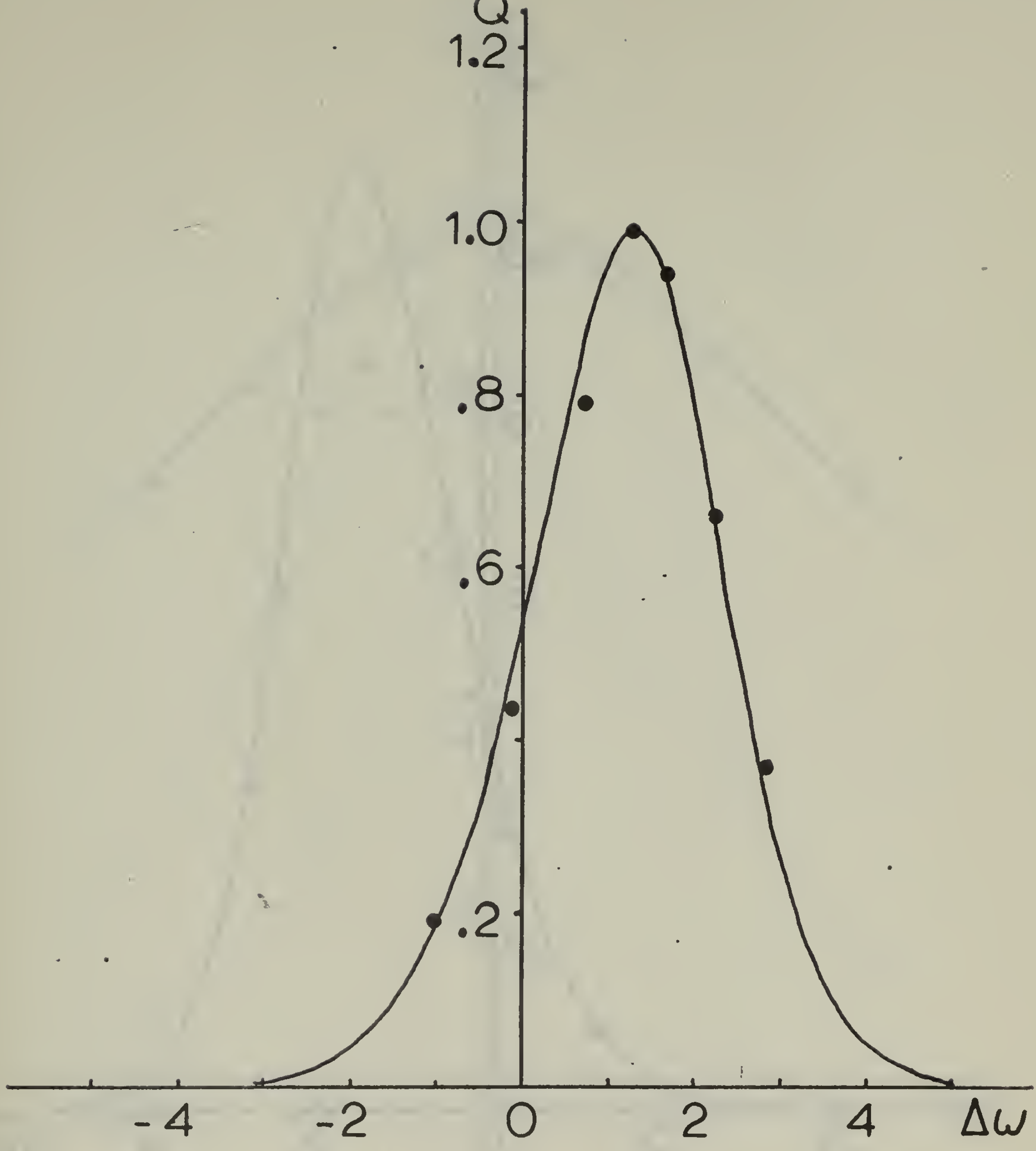
Q-curves for the third harmonic, M = 0.004,
———QCURC prediction,----QCURVES prediction
Experimental results are indicated by ●.

Figure C.7



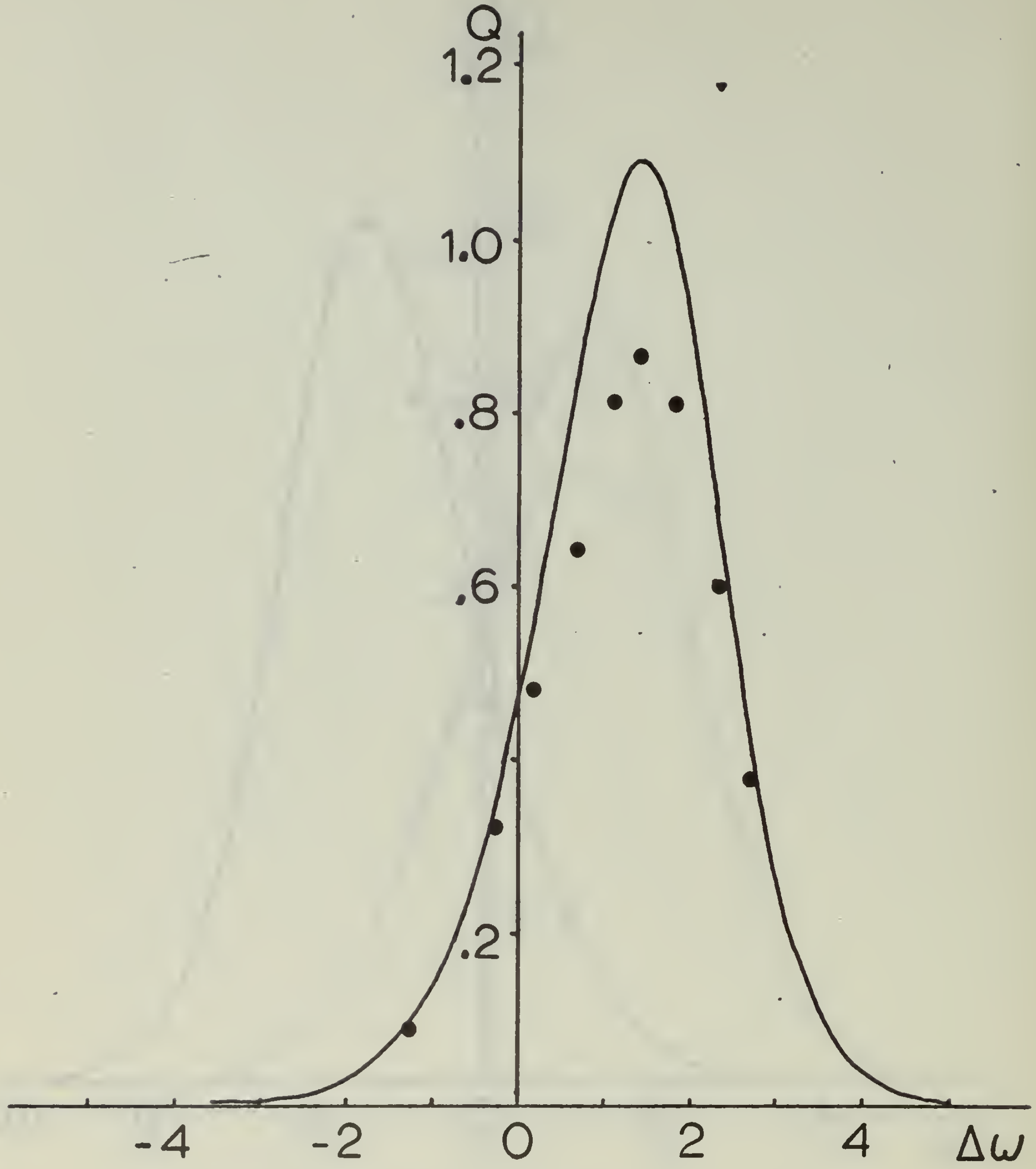
Q-curves for the fourth harmonic, M = 0.004,
____QCURC prediction,----QCURVES prediction
Experimental results are indicated by ●.

Figure C.8



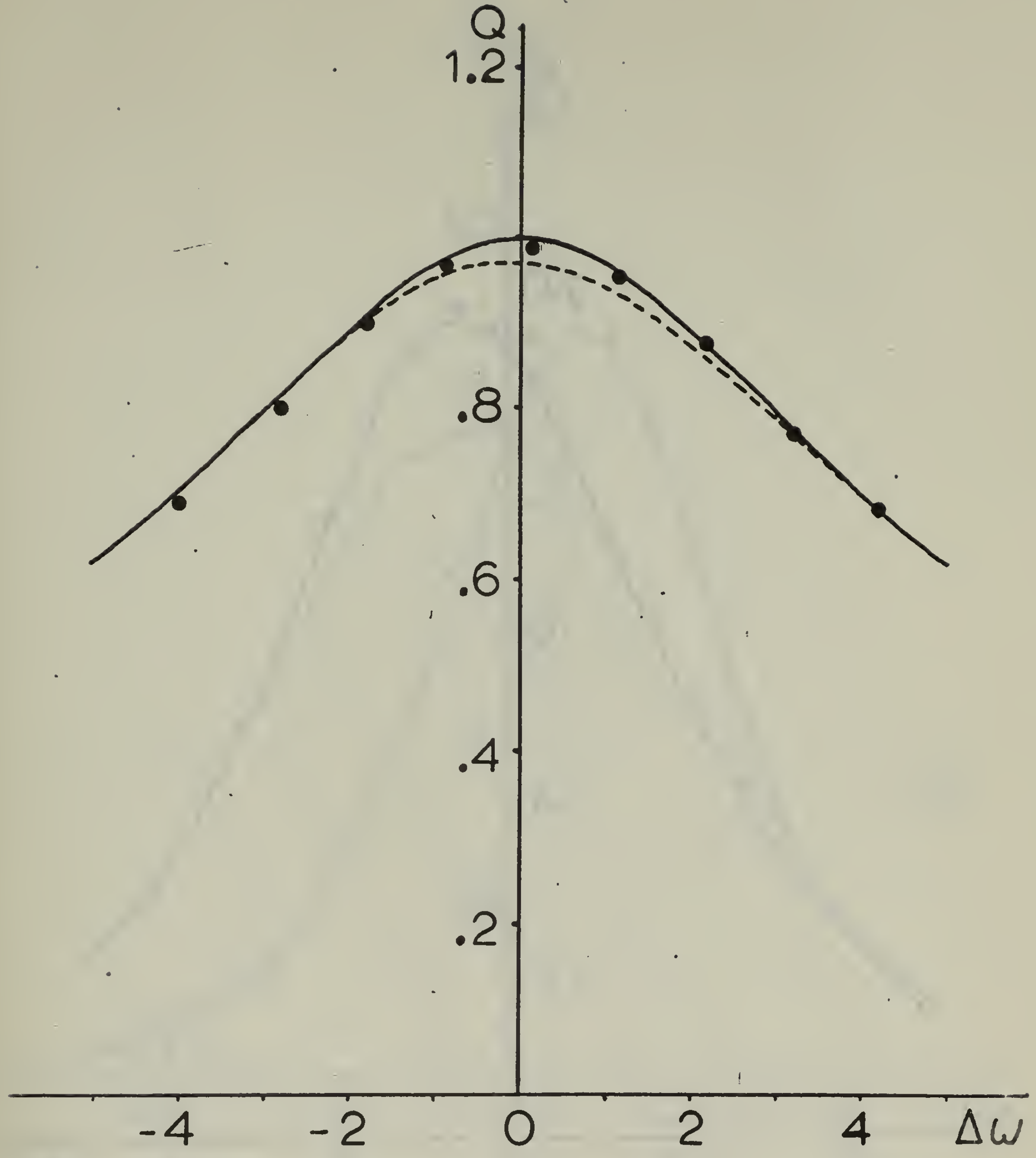
Q-curves for the fifth harmonic, M = 0.004,
____QCURC and QCURVES prediction
Experimental results are indicated by ●.

Figure C.9



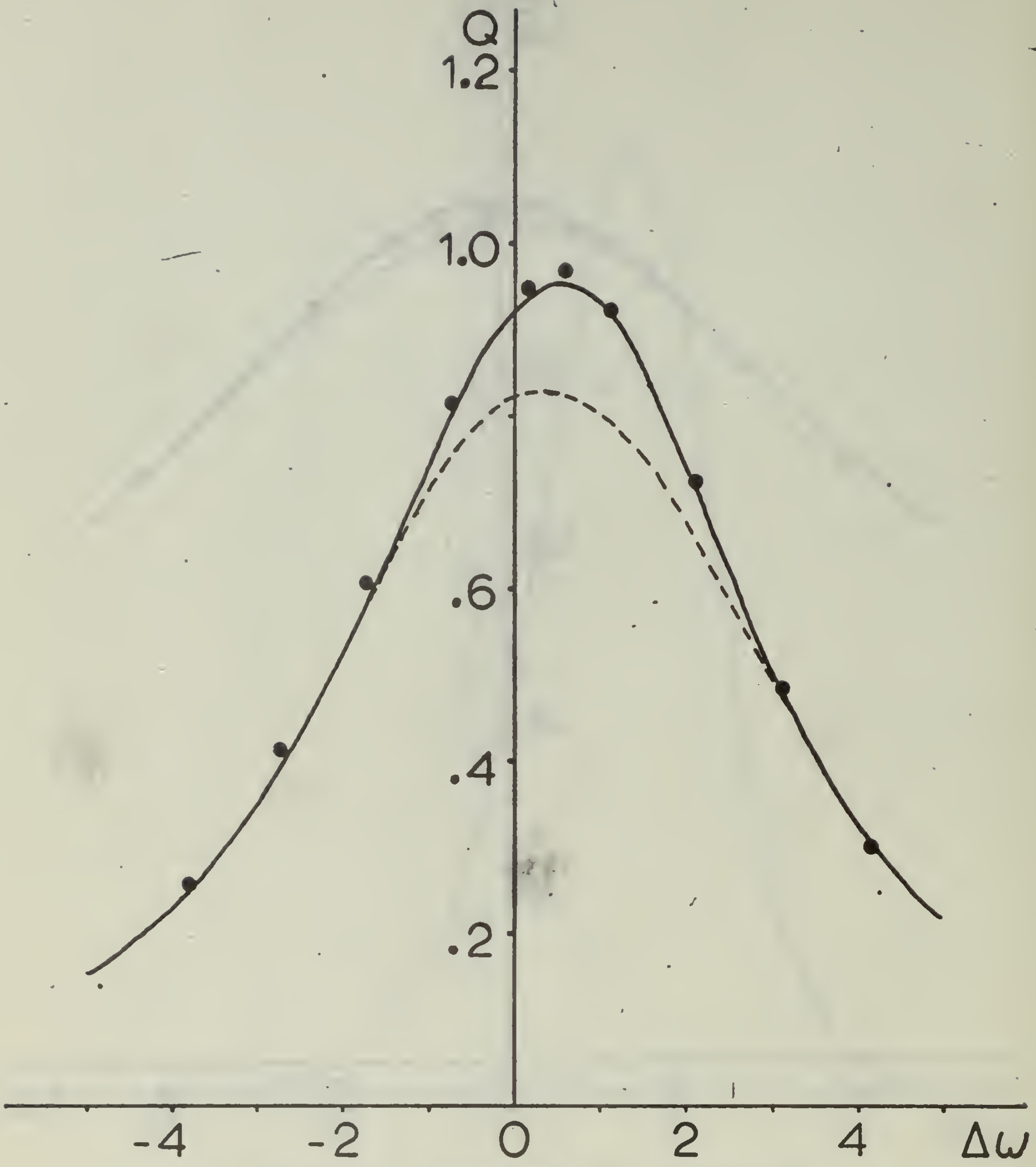
Q-curves for the Sixth harmonic, M = 0.004
_____QCURC and QCURVES prediction
Experimental results are indicated by ●.

Figure C.10



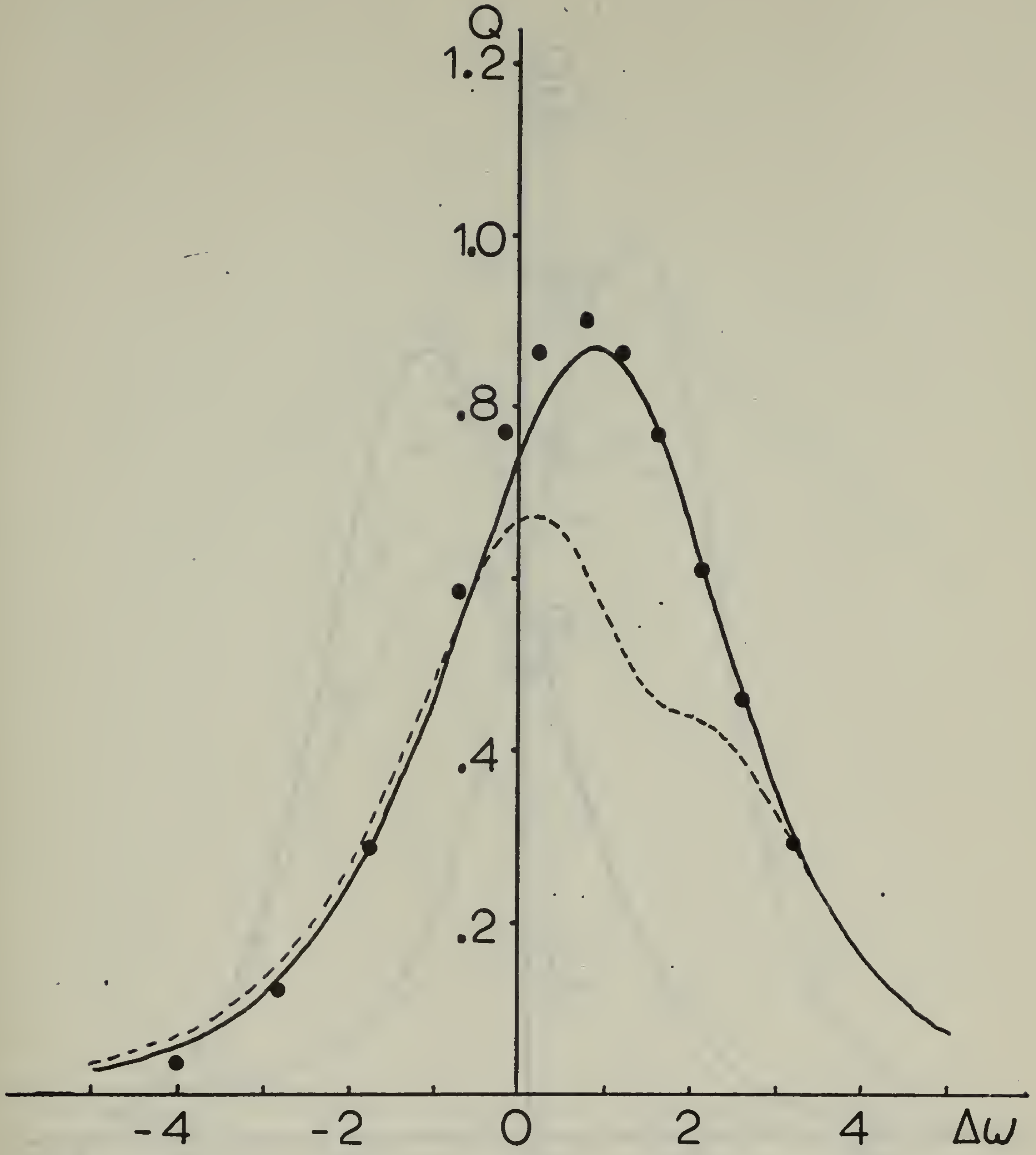
Q-curves for the fundamental, M = 0.005,
———QCURC prediction,----QCURVES prediction
Experimental results are indicated by ●.

Figure C.11



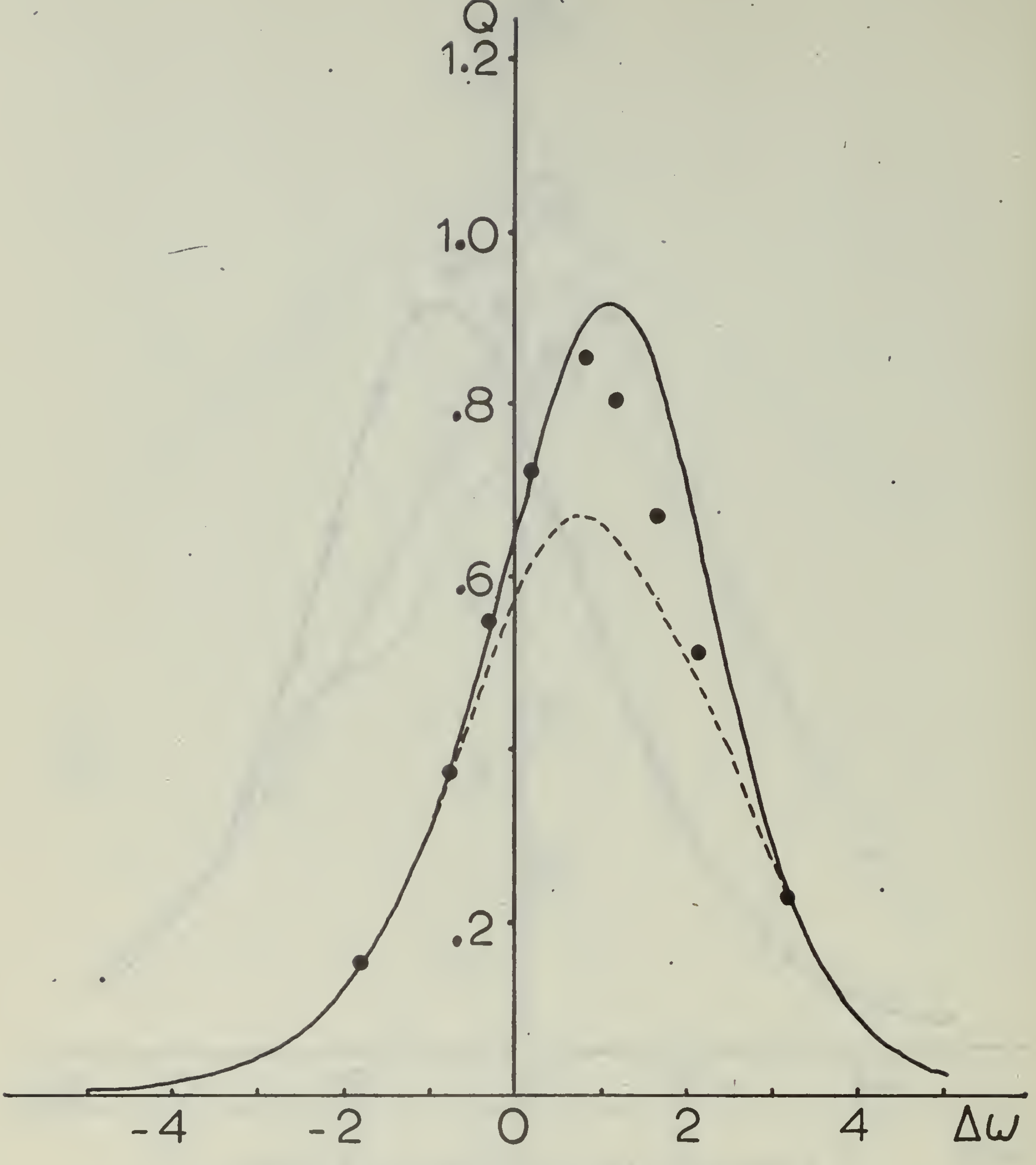
Q-curves for the second harmonic; M = 0.005,
_____QCURC prediction,----QCURVES prediction
Experimental results are indicated by ●.

Figure C.12



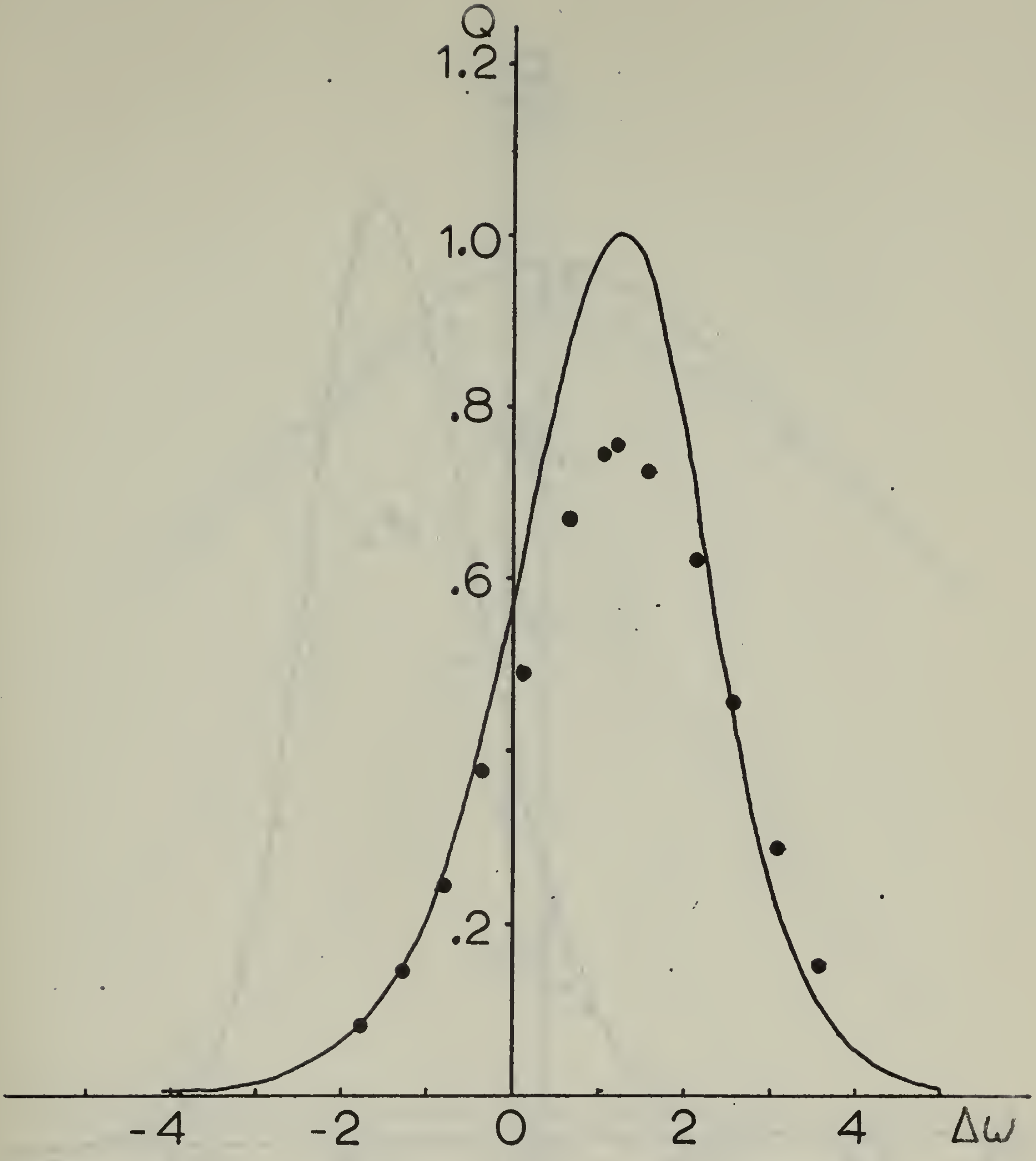
Q-curves for the third harmonic, M = 0.005,
_____QCURC prediction,---QCURVES prediction
Experimental results are indicated by ●.

Figure C.13



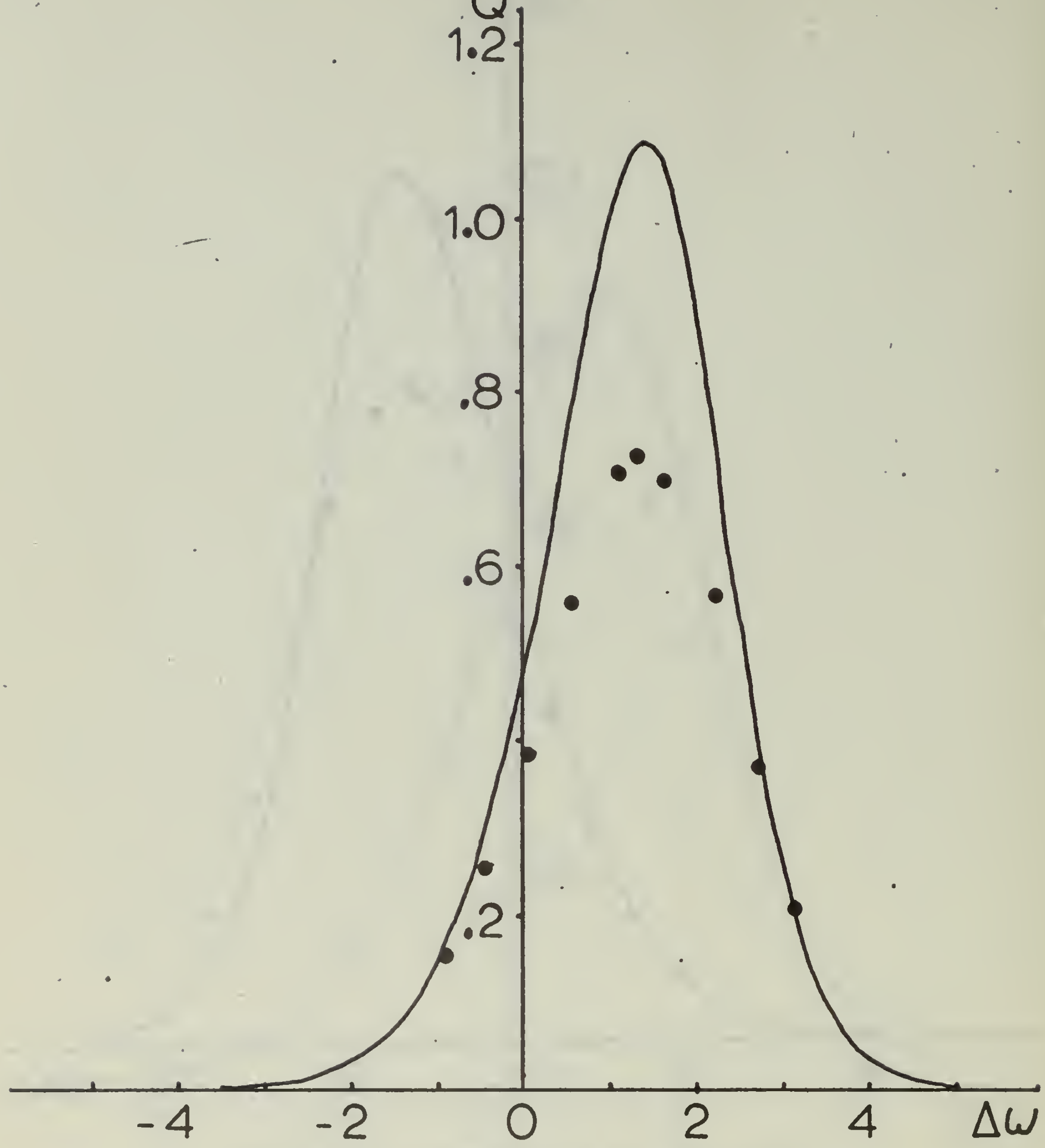
Q-curves for the fourth harmonic, M = 0.005,
____QCURC prediction,---QCURVES prediction
Experimental results are indicated by ●.

Figure C.14



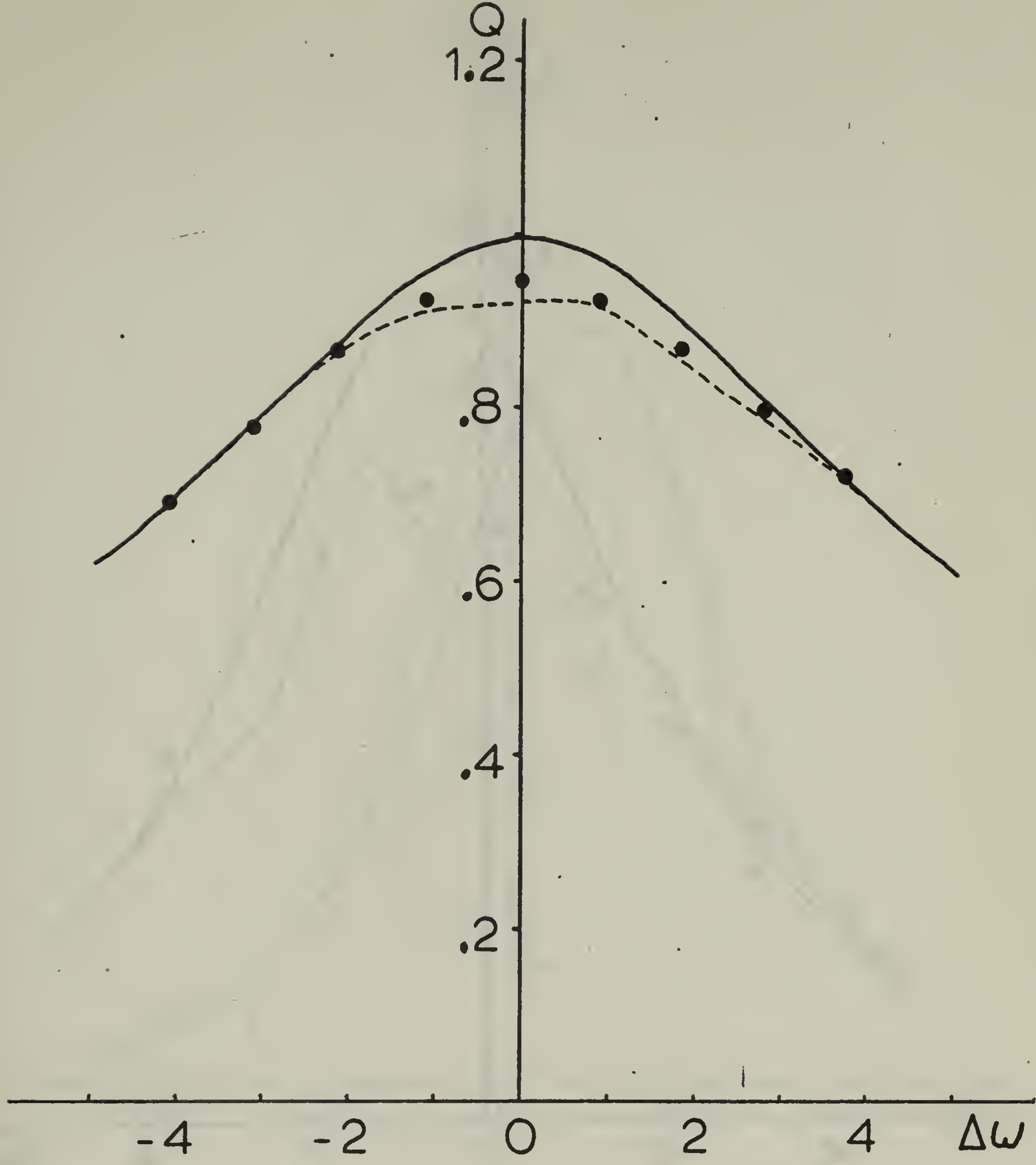
Q-curve for the fifth harmonic, M = 0.005,
____QCURC and QCURVES prediction
Experimental results are indicated by ●.

Figure C.15



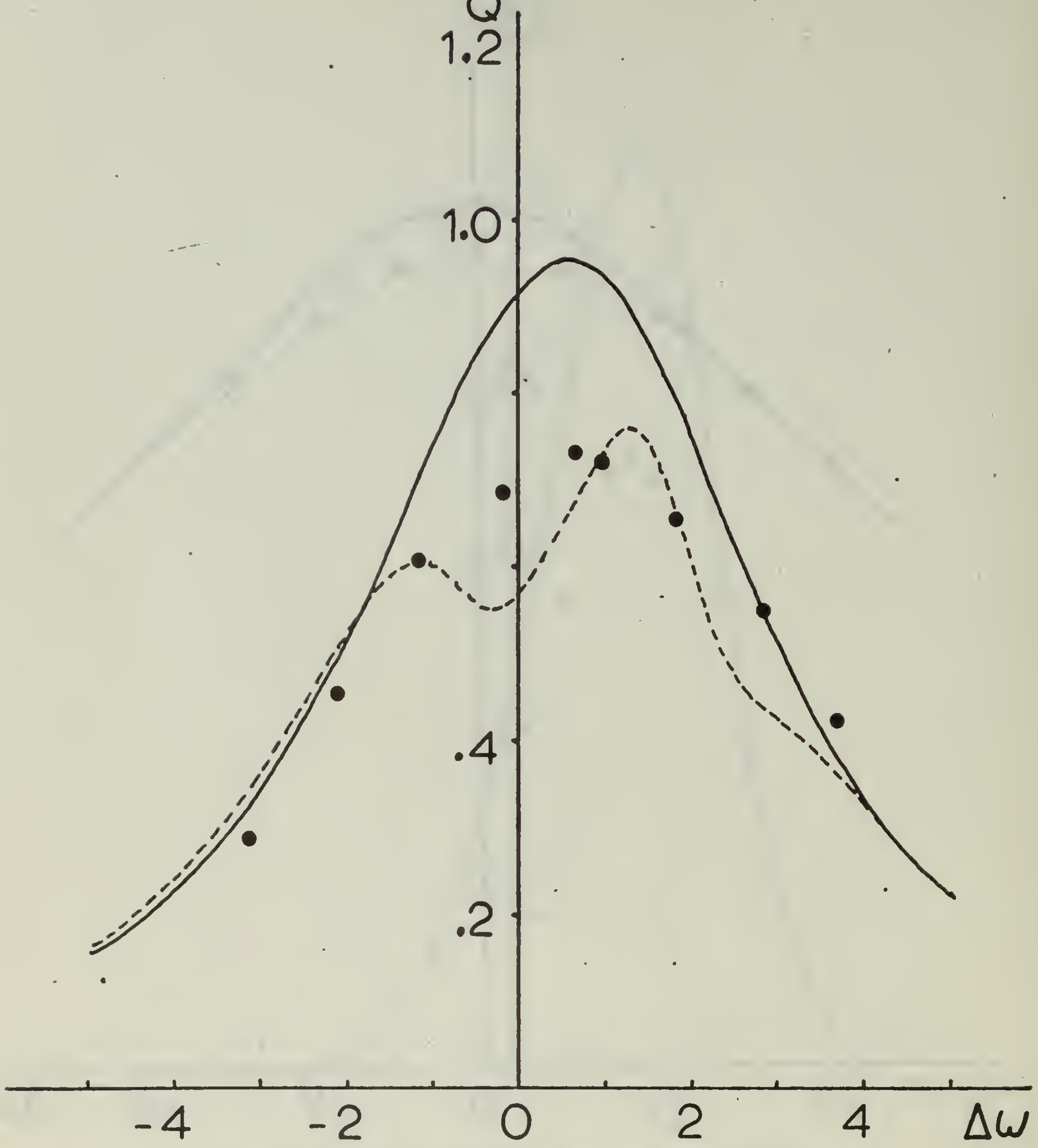
Q-curve for the sixth harmonic, M = 0.005,
____QCURC and QCURVES prediction
Experimental results are indicated by ●.

Figure C.16



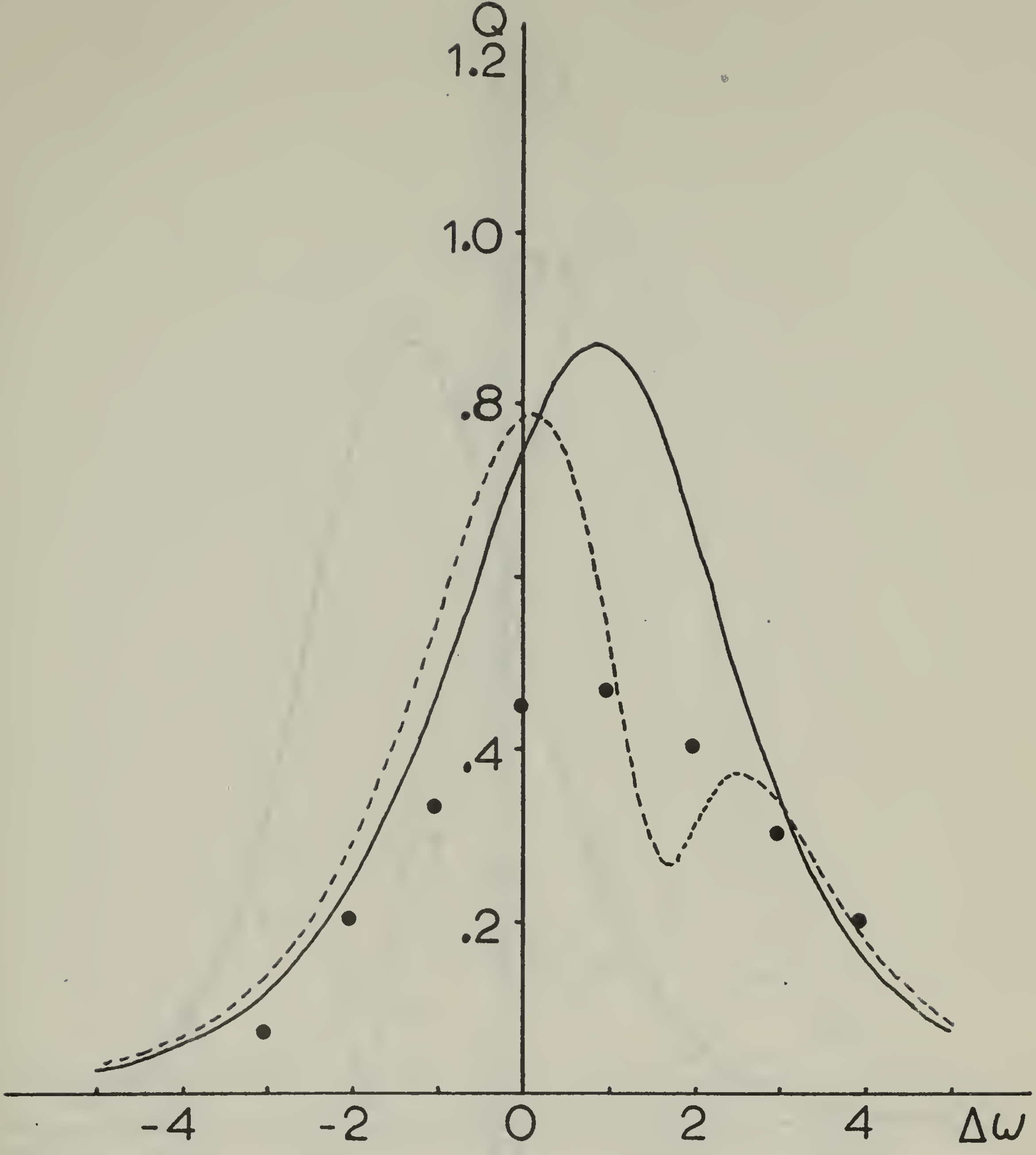
Q-curves for the fundamental, M = 0.009
—— QCURC prediction, --- QCURVES prediction
Experimental results are indicated by ●.

Figure C.17



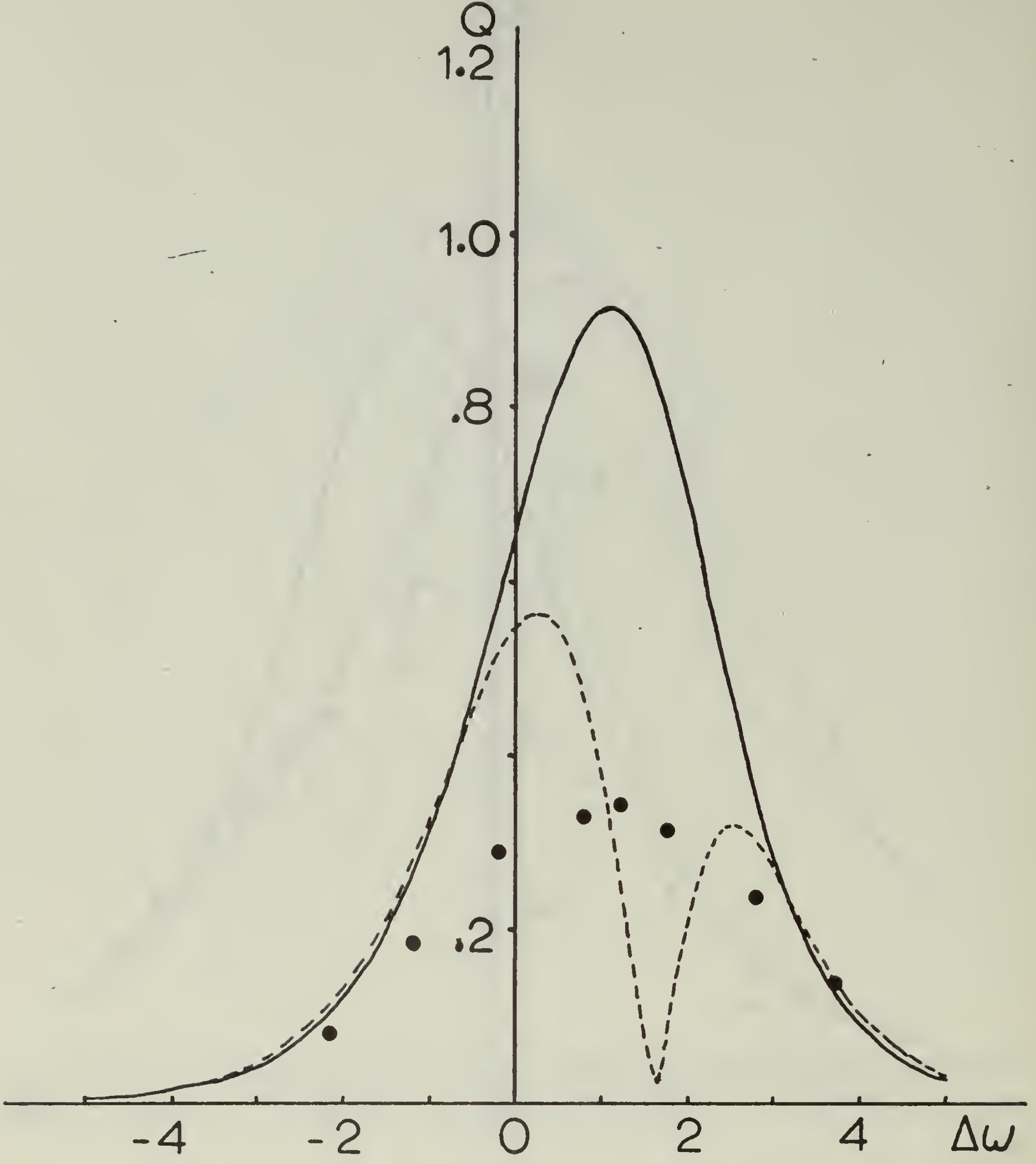
Q-curves for the second harmonic, M = 0.009,
_____QCURC prediction,----QCURVES prediction
Experimental results are indicated by ●.

Figure C.18



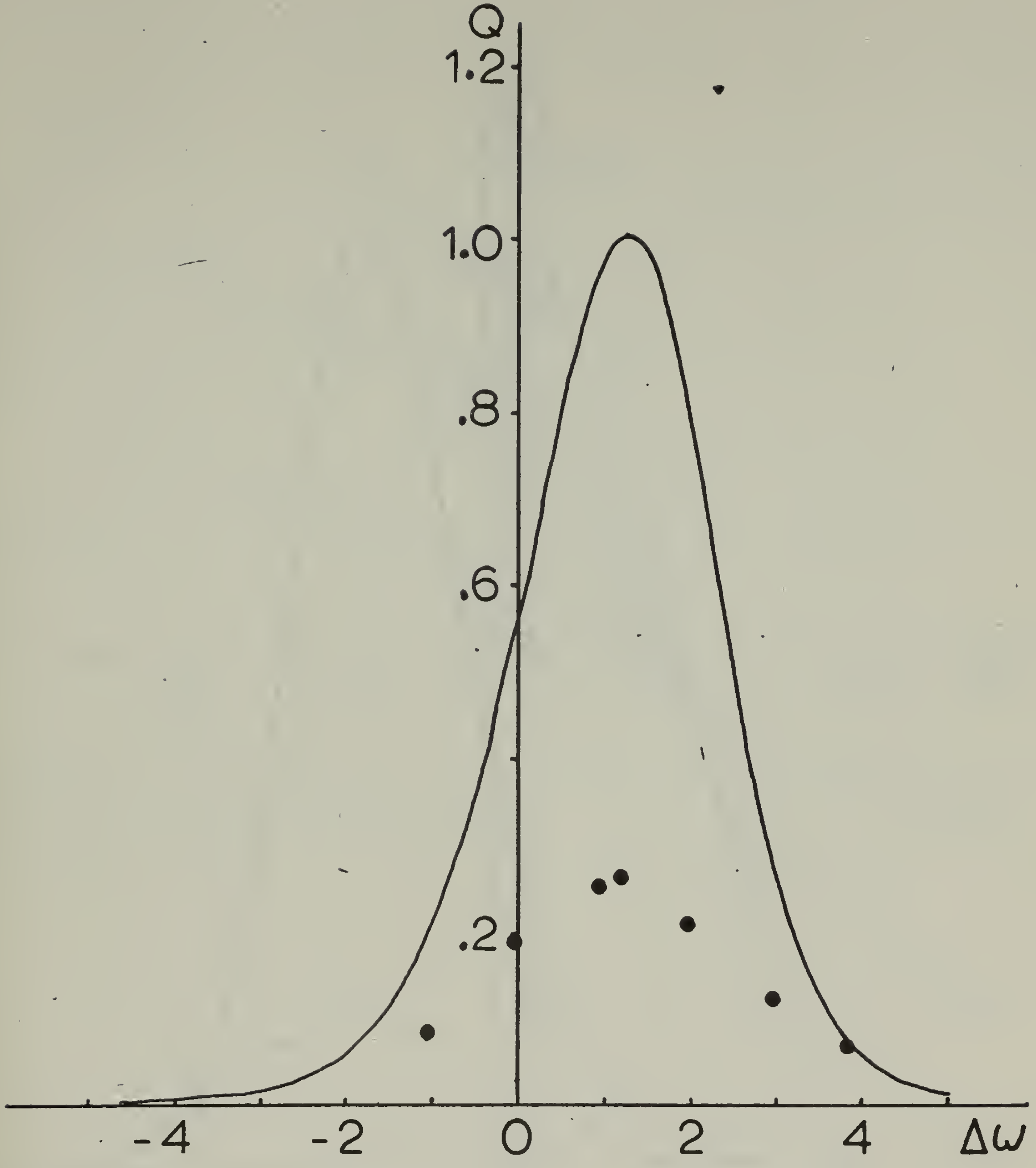
Q-curves for the third harmonic, M = 0.009,
_____QCURC prediction,----QCURVES prediction,
Experimental results are indicated by ●.

Figure C.19



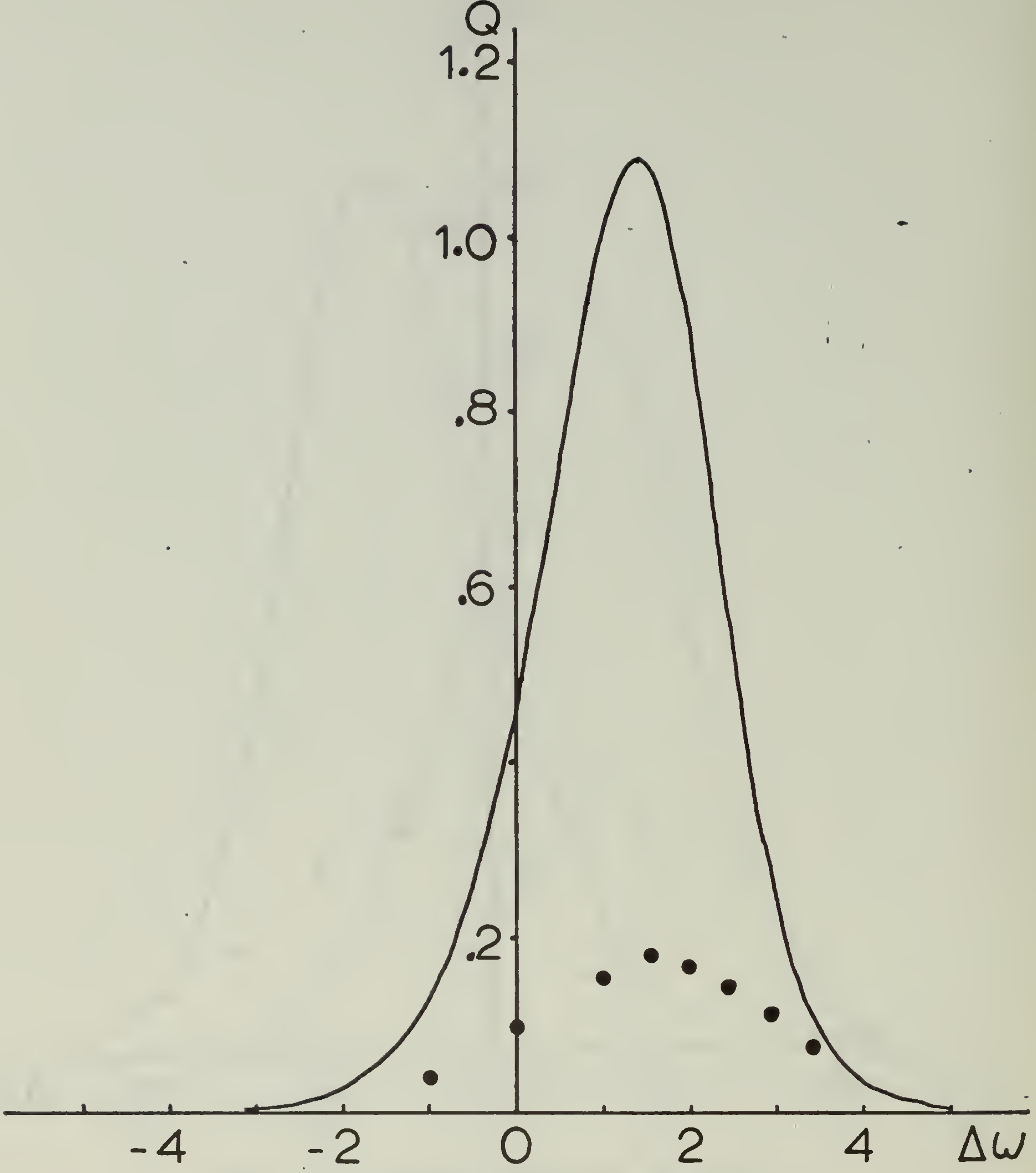
Q-curves for the fourth harmonic, M ≑ 0.009, ____QCURC prediction,---QCURVES prediction Experimental results are indicated by ●.

Figure C.20



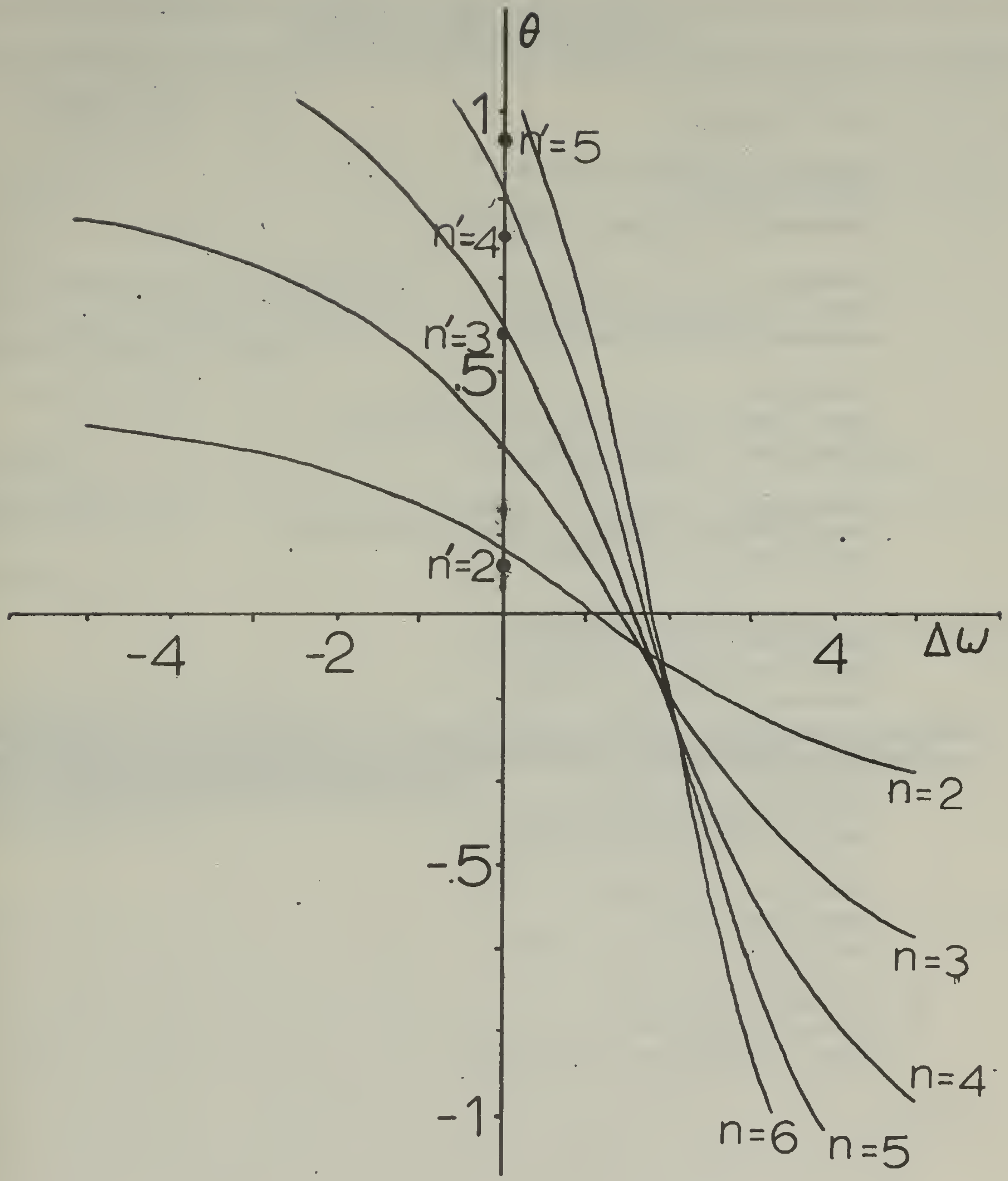
Q-curve for the fifth harmonic, M = 0.009,
____QCURC and QCURVES prediction
Experimental results are indicated by ●.

Figure C.21



Q-curve for the sixth harmonic, M = 0.009,
____QCURC and QCURVES prediction
Experimental results are indicated by ●.

Figure C.22



Phase dependence of the various harmonics on $\Delta\omega$ as predicted by PHAMPC. n = harmonic number, ● = data from FOUANAL for n'-th harmonic

Figure 23

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13. ABSTRACT

The Coppens-Sanders perturbation solution for the one-dimensional non-linear acoustic wave equation with dissipative term describing the viscous and thermal energy losses encountered in a rigid walled, closed tube with large length-to-diameter ratio was extended to include sixth order terms. The solution was then investigated to determine the region of validity. Computer programs were written to evaluate and graph the resulting waveforms. Available experimental results were compared with the theoretical predictions and good correlation was found to exist in the region of low Mach numbers. This agreement was found to gradually deteriorate as the Mach number was increased. A Fourier synthesis approach is also presented and the leading terms of the first ten harmonics are derived.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Finite-Amplitude						
Standing Waves						
Perturbation Approach						
Fourier Synthesis Approach						
Rigid-Walled Cavity						

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